Aspects of AdS/CFT:

Conformal Deformations and the Goldstone Equivalence Theorem

by

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Abstract

The AdS/CFT correspondence provides a map from the states of theories situated in AdS_{d+1} to those in dual conformal theories in a d-dimensional space. The correspondence can be used to establish certain universal properties of some theories in one space by examining the behave of general objects in the other. In this thesis, we develop various formal aspects of AdS/CFT.

Conformal deformations manifest in the AdS/CFT correspondence as boundary conditions on the AdS field. Heretofore, double-trace deformations have been the primary focus in this context. To better understand multitrace deformations, we revisit the relationship between the generating AdS partition function for a free bulk theory and the boundary CFT partition function subject to arbitrary conformal deformations. The procedure leads us to a formalism that constructs bulk fields from boundary operators. We independently replicate the holographic RG flow narrative to go on to interpret the brane used to regulate the AdS theory as a renormalization scale. The scale-dependence of the dilatation spectrum of a boundary theory in the presence of general deformations can be thus understood on the AdS side using this formalism.

The Goldstone equivalence theorem allows one to relate scattering amplitudes of massive gauge fields to those of scalar fields in the limit of large scattering energies. We generalize this theorem under the framework of the AdS/CFT correspondence. First, we obtain an expression of the equivalence theorem in terms of correlation functions of creation and annihilation operators by using an AdS wave function approach to the AdS/CFT dictionary. It is shown that the divergence of the non-conserved conformal current dual to the bulk gauge field is approximately primary when computing correlators for theories in which the masses of all the exchanged particles are sufficiently large. The results are then generalized to higher spin fields.

We then go on to generalize the theorem using conformal blocks in two and four-dimensional CFTs. We show that when the scaling dimensions of the exchanged operators are large compared to both their spins and the dimension of the current, the conformal blocks satisfy an equivalence theorem.

Advisor: Jared Kaplan

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Dedication

I dedicate this thesis to my parents, John and Davie Sue. Without my father's inspiration and guidance as the outstanding physicist and father he is and without my mother's unwavering support, infinite interest in my work, and patience through my periods of struggle, I could not conceivably have written this thesis or achieved even half of what I have today. I owe them everything and hope this thesis is the beginning of a return on their investment.

Contents

Abstract **ii**

Acknowledgments **iv**

List of Figures **xi**

1 Introducing AdS/CFT **1**

1.1 Anti-de Sitter space 3

1.2 Conformal fields theories 5

1.3 AdS/CFT correspondence 6

1.4 Chapter structure 7

2 Developing the Formalism: Conformal Deformations in AdS/CFT **8**

2.1 Introduction 8

2.2 Summary of results 12

2.3 Boundary conditions and classical
bulk fields 16

2.4 Bulk and boundary partition functions redux 22
2.5 Bulk correlation functions and the lift formalism 29
2.5.1 Correlating 29
2.5.2 Smearing 33
2.5.3 Lifting 34
2.6 Dilatation spectrum and RG flow 38
2.7 Discussion 42

3 Applying the Formalism: the Goldstone Equivalence Theorem 44

3.1 Introduction 44
3.2 Schwinger-Dyson Relations 53
3.3 ET in Flat Space 56
3.3.1 Proof for External Legs with Large Momenta 56
3.3.2 The Connection of the Exchange Operator to the ET 58
3.4 ET in AdS 61
3.4.1 AdS Wave Functions 62
3.4.2 The AdS ET 65
3.4.3 Implications for Correlation Functions of J_μ 73
3.5 Higher Spin AdS ET 76
3.5.1 AdS ET via Analysis of Wave Functions 78
3.5.2 Spin-l AdS ET at the Boundary 82

3.5.3 Generalization to CFT ET 83

3.6 Equivalence Theorem in CFTs 85

3.6.1 Two Dimensional Warm-Up 85

3.6.1.1 Spin-1 Current 87

3.6.1.2 Spin-3 Current 93

3.6.2 Higher Dimensions 97

3.6.2.1 Review of Index-Free Formalism of [1] and [2] 97

3.6.2.2 Spin-1 Current 102

3.7 Discussion 110

A Classical fields with multi-trace deformations 113

B Spontaneously Broken Gauge Lagrangian 117

C Review of the Schwinger-Dyson Equations 119

D Review of AdS Wave Functions 122

E Review of the LSZ Formula 126

F Review of Conformal Blocks and Partial Waves 128

G Details and Definitions 131

G.1 Partial Wave Definitions 131

G.2 Scalar Functions 132

H Applying Conservation **134**

I Full Result of Spin-1 Divergence **136**

Bibliography **142**

Vita **156**

List of Figures

1.1 Depiction of AdS 4

2.1 Witten diagram for bulk two-point function 30
2.2 1PI diagrams 31

3.1 Diagram of Goldstone equivalent theorem 45
3.2 Diagram of conformal Goldstone equivalence theorem 52
3.3 Four point scattering diagram 59
3.4 Four point scattering to leading order 60
3.5 AdS four point scattering diagram 68

Chapter 1

Introducing AdS/CFT

Physics has historically followed a reductionist philosophy, looking to shorter distances and more microscopic descriptions of nature to model systems whose aggregate behavior reproduces what the previous generation studied. Quantum mechanics was developed when classical methods failed at the atomic scale. Probing shorter distances requires greater energies, and sufficiently great energies demand the consideration of special relativity and permit the creation of new particles. This simultaneous need for quantum mechanics, special relativity, and the non-conservation of particle number led to the formal development of quantum field theory, a framework in which particles are modeled as excitations in fields[1] satisfying the canonical algebraic relations central to quantum mechanics.

Using the language of quantum field theory, the theoretical description of nearly

[1]These are mathematical objects that take on a value at each point in space-time.

all known physical phenomena has culminated in the Standard Model of particle physics. However, gravity is conspicuously absent from this description. If any object probes distances on the order of the Planck length, $l_p \sim 10^{-35}m$, it will inevitably collapse into a black hole since the object would require an energy on the order of $\frac{\hbar c}{l_p} \sim 10^9 J$, the Schwarzschild radius of which is the Planck length. Thus, the Planck scale designates the distance or energy scale at which gravity must be couched in a quantum framework, but this scale poses inherent obstructions to the usual language of quantum field theory because of the formation of black holes.

The AdS/CFT correspondence has emerged over the last two decades as our best means of understanding quantum gravity. Originally born as a mapping between states in a string theory living in a $d+1$ dimensional anti-de Sitter space (AdS_{d+1}) with compactified extra dimensions and supersymmetric Yang-Mills theories in a d dimensional space [3–5], the correspondence is now understood broadly as a duality between theories, most notably quantum gravity, in an AdS background and local conformal field theories (CFTs) that are defined at the infinite reaches of AdS. Such a duality, in which the content in a space manifests equivalently at its boundary, constitutes a holographic principle.

1.1 Anti-de Sitter space

Anti-de Sitter is a maximally symmetric space with a constant negative curvature, R, that arises as a solution to Einstein's field equations with a negative cosmological constant, Λ. It may possess a metric with either a Lorentzian structure, in which notions of time take on meaning and we asymptotically recover Minkowski space as $R \to 0$, or a Euclidean signature, in which there is no notion of time and the metric is positive definite. It is useful to visualize AdS$_{d+1}$ by embedding it in a d+2 dimensional flat space with coordinates in which the additional dimension is time-like. Lorentzian AdS is then the solution to

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2, \tag{1.1}$$

which appears as a cylinder in the embedding space; Euclidean AdS is the surface satisfying

$$X_{d+1}^2 - \sum_{i=1}^{d+1} X_i^2 = R^2, \tag{1.2}$$

appearing as hyperboloids in the embedding space. From Eq.(1.1), it follows that the isometry group of Lorentzian AdS$_{d+1}$ is SO(2,d); from Eq.(1.2), it follows for the Euclidean signature that the isometry group is SO(1,$d+1$).

"Global coordinates" is a particularly useful coordinate system that acts as a generalization of spherical coordinates to AdS. The metric for Euclidean (Lorentzian AdS$_{d+1}$) in global coordinates is

$$ds^2 = \frac{1}{\cos^2(\rho/R)} \left(dt^2 \pm d\rho^2 \pm R^2 \sin^2(\rho/R) d\Omega_{d-1}^2 \right), \tag{1.3}$$

where $\{\Omega_i\}$ contains polar coordinates. As a (pseudo-)Riemannian manifold, AdS should locally appear flat. Indeed, for $\rho \ll R$, Eq.(1.3) reproduces the appropriate flat-space metric in spherical coordinates.

It is conventional to choose units such that the curvature scale is set to $R = 1$. In this scale, the radial coordinate is restricted to $\rho \in [0, 2\pi]$, such that $\rho = 2\pi$ corresponds to the boundary at infinity. Consequently, global coordinates permit a compact representation of all of AdS. Depictions of Lorentzian and Euclidean AdS in global coordinates is given in Fig.(1.1).

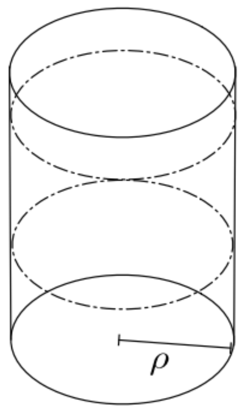

Figure 1.1: A depiction of AdS in global coordinates. The cylindrical structure of AdS endows the boundary with an $\mathbb{R} \times S_{d-1}$ topology.

The "Poincaré patch" is another useful coordinate system that behaves like Cartesian coordinates with a warped dimension. The metric in the Poincaré patch for Euclidean (Lorentzian AdS_{d+1}), with $R = 1$, is given by

$$ds^2 = \frac{1}{z^2}\left(dt^2 \pm dz^2 \pm \sum_{i=1}^{d-1} dx_i^2\right). \tag{1.4}$$

For each slice of constant z, Eq.(1.4) makes manifest the SO(d) (SO(1,$d-1$)) subgroup of the Euclidean (Lorentzian AdS_{d+1}) isometry group. The Poincaré patch does not span all of AdS, but rather it covers a wedge with a "full" boundary located at $z = 0$ and a zero-sized boundary at $z \to \infty$.

1.2 Conformal fields theories

Conformal field theories (CFTs) are, loosely, quantum field theories that lack a scale, which is any quantity that sets a preferred energy near which systems exhibit a special behavior. Giving a particle a mass, for instance, specifies a scale: the minimum energy necessary to create the particles. Theories lacking any such scale enjoy a dilatation symmetry, the action of which simply rescales the energy of a system described by the theory. CFTs are typically constructed with initially only scale invariance in mind, but, arguably, inevitably exhibit the full conformal symmetry, which consists of dilatations as well as the so-called special conformal transformations, which translate a system then invert its distance with respect to the origin of the coordinate system [6–8]. Explicitly, the conformal group for a theory in a d-dimensional Euclidean (Lorentzian) space is SO(1,$d+1$) (SO(2,d)), the same as the isometries of their respective AdS_{d+1} counterparts.

1.3 AdS/CFT correspondence

It should not be surprising that the AdS/CFT correspondence is a duality since the two theories exhibit the same symmetries. A duality is not a physical equivalence, but simply a mathematical prescription with which structures or concepts may be translated into other structures or concepts. For example, the displacement of a classical harmonic oscillator obeys the same equations as the charge in an LC circuit, permitting an initial-value problem in one to be mapped to the other. While there is a physical distinction, the mathematical operations that can be done to one unambiguously translate to the other.

In the case of the AdS/CFT correspondence, a conformal operator, generally constructed from the local fields of a CFT, is identified as the dual to the asymptotic behavior of a field in AdS as the position at which the field is evaluated is taken to the boundary. For a scalar field in the Poincaré patch, this is the statement that

$$\mathcal{O}(x) = \lim_{z \to 0} z^{-\Delta} \phi(x, z), \tag{1.5}$$

where $\mathcal{O}$ is the conformal operator, Δ is the scaling dimension of $\mathcal{O}$, and ϕ is the AdS dual to $\mathcal{O}$. The factor of $z^{-\Delta}$ projects out the vanishing behavior of the field as it is taken to the boundary. This identification leads to a correspondence in the values of observables, such as correlation functions, between the AdS and CFT theories.

1.4 Chapter structure

Few conformal theories exist in nature. However, many quantum field theories may be constructed as broken CFTs by adding a "deformation" to an otherwise conformal theory. The AdS/CFT correspondence has been primarily considered in the presence of at most double-trace deformations, leaving room for the formal development of techniques to handle multi-trace deformations. We develop these techniques in Chapter 2 and apply them to holographic RG flow.

The Goldstone equivalence theorem relates processes involving longitudinally polarized massive gauge bosons (vector particle) to the those of scalars when the energies involved are large compared to the gauge boson's mas. In Chapter 3, the equivalence theorem in AdS is developed and used to explicate the relationship between non-conserved currents and scalars in the CFT dual.

The relevant literature reviews for each topic will be covered at the beginning of each chapter.

Chapter 2

Developing the Formalism: Conformal Deformations in AdS/CFT

2.1 Introduction

The AdS/CFT correspondence has been understood via two dictionaries:

1. Taking AdS correlation functions to the boundary and extracting the leading order behavior to recover correlators of dual operators constructed from a local CFT [9, 10],

$$\langle \mathcal{O}\mathcal{O} \dots \rangle = \lim_{z \to 0} \langle z^{-\Delta} \phi(z) z^{-\Delta} \phi(z) \dots \rangle$$

and

2. Evaluating the on-shell AdS partition function as a functional of a boundary source, ϕ_b, and computing CFT correlators in the usual way [4, 5],

$$Z_{\mathrm{CFT}}[\phi_b] = Z_{\mathrm{AdS}}[\phi_b].$$

The dictionaries have been shown to be equivalent in the presence of bulk interactions [11], the intuition being that interactions turn off near the boundary [12], rendering the on-shell description adequate.

The AdS/CFT correspondence also provides a holographic means to understand the renormalization group flow of CFTs as the classical evolution of dual bulk theories in the radial direction of AdS [13–25]. Explicitly, the radial coordinate is interpreted as the renormalization scale. Efforts to approach RG flow via entropy-esque quantities and H-theorem type constraints have lead to the a-, F-, and c- theorems [23, 26–28], and their associated geometric formulations [21, 24, 25].

While few conformal theories appear in nature, deformations can be added to certain ones to more closely reproduce physical theories. Naturally, we must consider what becomes of the AdS/CFT correspondence when conformality at the boundary is spoiled since, by construction, interactions do not turn off near the boundary. Conformal deformations were first examined in the context of AdS/CFT in [29], and recent work in conformal dominance [30, 31] invites their continuing presence. The role of boundary conditions in AdS theories was examined by [32], and the connection between boundary conditions and CFT deformations was made explicit by [33, 34].

Double-trace deformations have been the predominant focus in the context of AdS/CFT [29, 33–36]. While demonstrative of many salient features of the deformed correspondence, most double-trace techniques, in which the relationship between the bulk and boundary field remains linear while approaching the boundary, do not manifestly apply to more general deformations. It is also unclear from the literature whether we are instructed to evaluate the AdS partition function on-shell in the usual manner when employing the second dictionary to compute correlators. By this we mean computing the bulk field as the classical solution to the field equations in the presence of a boundary source, ϕ_b. Intuition says 'no' as this would omit quantum effects from our correlators.

In the first part of this paper, §2.3 and §2.4, we aim to clarify the ambiguities in handling multi-trace deformations and establish a framework that makes manifest the equivalency of the two dictionaries subject to these deformations. This is achieved by deriving the explicit relationship between the generating bulk partition function and the dual CFT partition function with deforming Lagrangian $W[\mathcal{O}]$.

This framework then leads to what we call a lift formalism in §2.5.3 that is akin to the effort to construct local bulk observables via nonlocal smearing of the boundary [37–42]. The formalism diverges from smearing at the level of operators by providing a nonlinear map between the boundary and the bulk, but similarly achieves the goal of computing bulk correlators from boundary data. This is done to provide another chapter in the AdS/CFT dictionary and to offer an alternative means of computing Witten diagrams of bulk correlators with boundary deformations, the former seeming particularly useful if boundary data is to be used to understand bulk phenomena.

Capitalizing on some results from the lift formalism, we examine in §2.6 the RG flow triggered by general deformations by interpreting the location of the UV brane used to regulate the AdS theory as a renormalization scale. Specifically, we compute the scale-dependence of the conformal dimension of the CFT. We will then briefly comment on potential applications to the recent work of conformal dominance/the truncated Hamiltonian space approach [30, 31, 43–45].

We work in the Poincaré patch of Euclidean AdS_{d+1} with

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \sum_{i=1}^{d} dx_i^2\right) \tag{2.1}$$

and $R_{AdS} = 1$. We consider only CFTs dual to free theories in the bulk. We employ the compact notation $\bar{d} = \frac{d}{2\pi}$ for integral measures. We will consider only scalar fields herein, and, where applicable, we will consider a general scaling dimension, $\Delta \geq \frac{d}{2} - 1$, of the operator dual to the bulk field; however, with the goal of examining RG flow in mind, we will usually restrict the scaling dimension in the UV to $\Delta = \Delta_- \leq \frac{d}{2}$.

We first offer a convenient summary of the results of this chapter.

2.2 Summary of results

The generating partition function from which bulk correlators can be computed is

$$
\begin{aligned}
&Z_{AdS}[J] = \exp\left[\int d^d x \int dz\, \frac{1}{2} J(x,z)\phi_{cl}(x,z)\right] \\
&\times \int \mathcal{D}\alpha \mathcal{D}\beta \exp\left[\int_{z=\epsilon} d^d x\, \left(-\nu\epsilon^{2\nu}\beta^2(x) + W[\alpha(x) + \epsilon^{2\nu}\beta(x) + \epsilon^{-\Delta_-}\phi_{cl}(x,\epsilon)]\right)\right. \\
&\left. + S_\partial[\alpha]\right]. \qquad (2.2)
\end{aligned}
$$

AdS/CFT correlators are generated by functionally differentiating Eq.(2.2) with respect to to the source J. Bulk fields scale to the boundary as $\phi \underset{z\to 0}{\to} \alpha z^{d/2-\nu} + \beta z^{d/2+\nu}$, and the classical, particular solution to the field equations is given by $\phi_{cl}(x,z) = \int d^d x' \int_{z'=0}^{\infty} G(x-x'; z, z') J(x', z')$, where G is the bulk-bulk propagator. $W[\mathcal{O}]$ is the deforming Lagrangian for the dual CFT ($\alpha \to \mathcal{O}$) with $\Delta_- = \frac{d}{2} - \nu$, $0 < \nu < 1$ as the conformal dimension of $\mathcal{O}$. From the perspective of the bulk, W is just a boundary term, rendering the bulk theory free. $z = \epsilon \ll 1$ is the location of the UV brane on which the bulk theory terminates that is used to regulate the AdS theory. S_∂ is a generic conformal action that generates dynamics for $\mathcal{O}$. The functional integral over α and the written dependence of S_∂ on α are formalities that simply instruct us to evaluate α as $\mathcal{O}$ in the CFT.

β is an auxiliary field, and integrating it out reproduces Witten's prescription for the boundary conditions:

$$\beta = \frac{1}{2\nu} W'[\alpha]. \tag{2.3}$$

The partition function given in Eq.(2.2) differs from what usually appears in the literature, where the "on-shell" action is taken to have the form $S \propto \int_{z=\epsilon} d^d x \, \alpha\beta$ to leading order in ϵ. In this paper, we argue that these leading terms are canceled and that second order effects must thus be considered to correctly yield Eq.(2.3) from the on-shell behavior for β.

It additionally follows from Eq.(2.2) that setting $\epsilon \to 0$, $J \to 0$, and $\phi_b \alpha \subset W[\alpha]$, yields

$$Z_{AdS}[\phi_b] = Z_{CFT}[\phi_b], \tag{2.4}$$

establishing the second line in Eq.(2.2) as a modified form of the CFT partition function and confirming the second dictionary in the presence of boundary deformations.

Constructing the AdS partition function from the CFT partition function as above is reminiscent of the use of smearing functions to construct AdS operators from their CFT duals. Smearing functions typically map a tower of operators to a bulk field. We offer a similar procedure that maps the operators in a different way, but recovers the ultimate goal of constructing AdS correlators from boundary ones. At the level of operators, we write

$$\phi(x,z) = \int d^d x' L_\alpha(x,x';z)\alpha(x') + 2\nu \int d^d x' L_\beta(x,x';z)\left(\beta(x') + \beta_0(x')\right), \tag{2.5}$$

where the lift kernels are given by

$$L_\alpha(x,x';z) = \int đ^d p \, \Gamma(1-\nu)\left(\frac{p}{2}\right)^\nu z^{d/2} I_{-\nu}(pz) e^{ip\cdot(x-x')} \tag{2.6}$$

$$L_\beta(x,x';z) = \int đ^d p \left[\frac{1}{2}\Gamma(\nu) I_{-\nu}(pz) - \frac{1}{\Gamma(1-\nu)} K_\nu(pz)\right]\left(\frac{p}{2}\right)^{-\nu} z^{d/2} e^{ip\cdot(x-x')}. \tag{2.7}$$

β_0 acts as a functional derivative when inserted into correlators:

$$\beta_0(x) = \frac{1}{2\nu}\frac{\delta}{\delta\alpha(x)}, \tag{2.8}$$

while β takes its usual form of Eq.(2.3).

These results indeed confirm that the bulk field cannot be computed directly as a classical functional of ϕ_b when using the second dictionary in the presence of general deformations. It must be computed in terms of α, which itself can only be computed as a classical functional of ϕ_b when computing correlators in the presence of, at most, double-trace deformations.

Using the formalism, we find two interesting results for particular boundary correlators in momentum space:

$$\langle W'[\alpha](-p)W'[\alpha](p) + W''[\alpha](p)\rangle = \frac{\Sigma(p)}{1-g(p)\Sigma(p)}, \tag{2.9}$$

and

$$\langle \alpha W'[\alpha]\rangle(p) = \frac{g(p)\Sigma(p)}{1-g(p)\Sigma(p)}. \tag{2.10}$$

Here, g is the *free* boundary propagator and Σ is the sum over two-point function 1PI diagrams at the boundary. Evidently, there is a strong connection between β

terms in the AdS/CFT dictionary and 1PI diagrams at the boundary. The generally non-vanishing value of the RHS of Eq.(2.10) indicates that the normal ordering one would naively apply to W' when treating it as a multitrace operator is instead applied to the full $\alpha W'$, meaning that W' generally contains both multitrace operators and additional, non-normal ordered operators. The connection to 1PI diagrams is thus not surprising since W' involves operators that appear in the OPE spectrum of the theory.

We go on to find the conformal dimension of a dual boundary operator depends on RG scale, μ, as

$$\Delta \underset{\mu\to\infty}{=} \left.\frac{-\mu\partial_\mu\left[\langle\phi\alpha\rangle(p,\mu)\right]}{\left[\langle\phi\alpha\rangle(p,\mu)\right]}\right|_{|p|=\mu}. \tag{2.11}$$

We are instructed to evaluate the correlators with $z = \mu^{-1}$ as a UV brane. Specifically, we find the bulk-boundary propagator and pull it to the UV brane on which the CFT sits.

It is useful to demonstrate the procedure by computing Δ for the well-understand example of a double-trace deformation, $\lambda\mathcal{O}^2$, with UV scaling dimension Δ_-. We find in momentum space near the boundary

$$\begin{aligned}&\left.\langle\phi\alpha\rangle(p,\mu^{-1})\right|_{|p|=\mu} \underset{\mu\to\infty}{=}\\ &-\frac{1}{2}\left[\frac{1}{1+\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\left(\frac{\mu}{2}\right)^{-2\nu}\frac{\lambda}{2}}\right]\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\left(\frac{\mu}{2}\right)^{d/2-\nu}.\end{aligned} \tag{2.12}$$

Inserting Eq.(2.12) into Eq.(2.11) reproduces the known result for double-trace deformations:

$$\Delta(\mu) = \Delta_- + \frac{\lambda}{\mu^{2\nu} + \frac{\lambda}{2\nu}}. \tag{2.13}$$

Eq.(2.13) has the simple interpretation that operators in a theory with double-trace deformations with scaling dimension Δ_- in the UV ($\mu \to \infty$) flow to operators with scaling dimensions $\Delta_+ = \Delta_- + 2\nu$ in the IR ($\mu \to 0$).

2.3 Boundary conditions and classical bulk fields

Conformal deformations manifest as boundary conditions in AdS. Witten originally proposed that a deformation of the form

$$S_W[\mathcal{O}] = \int d^d x\, W[\mathcal{O}], \tag{2.14}$$

where $\mathcal{O}$ is a generalized free field with scaling dimension Δ, should be interpreted as a boundary action in AdS. For dual bulk fields that scale to the boundary as

$$\phi(x,z) \underset{z\to 0}{\to} \alpha(x) z^{\Delta} + \beta(x) z^{d-\Delta}, \tag{2.15}$$

the boundary term becomes $S_W[\alpha]$. This leads to the following constraint on the asymptotic behavior[1]:

$$\beta(x) = \frac{1}{d-2\Delta}\frac{\delta S_W[\alpha]}{\delta \alpha(x)}. \tag{2.16}$$

[1]The coefficient $\frac{1}{d-2\Delta}$ is actually dependent on the normalization of $\mathcal{O}$. Here, we take $\alpha = \langle \mathcal{O} \rangle$.

This condition was argued to arise classically from an on-shell bulk theory ending on a UV brane at $z = \epsilon \ll 1$ whose action is given by [46]

$$S[\phi] = S_{bulk}[\phi] + S_{ct}[\phi] + S_W[\phi] = \int d^d x \int dz \sqrt{g}\, \frac{1}{2} \left\{ z^2 (\partial\phi)^2 - \left[\left(\frac{d}{2}\right)^2 - \nu^2 \right] \phi^2 \right\} + \int_{z=\epsilon} d^d x \sqrt{g} \mathcal{L}_{ct} + \int_{z=\epsilon} d^d x\, W[\epsilon^{-\Delta}\phi], \tag{2.17}$$

where

$$S_{ct} = \int_{z=\epsilon} d^d x\, \frac{1}{2} \epsilon \Delta \phi^2 \tag{2.18}$$

is a boundary counter term that ensures the convergence of the on-shell action for $\Delta \geq \frac{d}{2} - 1$ [47,48]. For $\Delta \leq \frac{d}{2}$, the counter term additionally specifies which of the two viable dual conformal theories sits at the boundary, $\Delta = \Delta_\pm = \frac{d}{2} \pm \nu, 0 \leq \nu \leq 1$. The mass-squared has been written in terms of $\nu \equiv \sqrt{\left(\frac{d}{2}\right)^2 - m^2}$ for later convenience.

Classically, Eq.(2.17) with $\mathcal{L}_{ct} = \frac{1}{2}\epsilon\Delta\phi^2$ leads to the differential equations

$$\left[\mathcal{D}_z^2 + \mathcal{D}_\partial^2\right] \phi = 0 \tag{2.19}$$

$$\left[B_0 + \delta B\right] \phi = 0, \tag{2.20}$$

where

$$\mathcal{D}_z^2 = z^{d+1} \partial_z \left[z^{-d+1} \partial_z \right] + \left[\left(\frac{d}{2}\right)^2 - \nu^2 \right], \tag{2.21}$$

$$\mathcal{D}_\partial^2 = z^2 \partial_\mu \partial^\mu, \tag{2.22}$$

$$B_0 = \epsilon\partial_z - \Delta|_{z=\epsilon} \tag{2.23}$$

$$\delta B = -\,\epsilon^{d-\Delta} W'[\epsilon^{-\Delta}\phi]\big|_{z=\epsilon}. \tag{2.24}$$

Note, the μ index runs over only the boundary coordinates. Eqs.(2.20)&(2.15) then imply, to leading order in ϵ,

$$(d-2\Delta)\beta = W'[\alpha], \tag{2.25}$$

which is, as promised, Eq.(2.16). For now, the classical arguments will be kept to leading order in ϵ; it will be shown later than we must consider second order effects to properly recover the condition in the quantum theory.

We employ the above cumbersome notation to provide a simple, general solution scheme to the classical bulk equations. If we solve the easier problem given by

$$\mathcal{D}_z^2 \mathcal{Z} = 0 \tag{2.26}$$

$$B_0 \mathcal{Z} = -\delta B \phi, \tag{2.27}$$

and write $\varphi \equiv \phi - \mathcal{Z}$, we can reduce our task to solving a theory with better known boundary conditions:

$$\left[\mathcal{D}_z^2 + \mathcal{D}_\partial^2\right] \varphi = -\mathcal{D}_\partial^2 \mathcal{Z} \tag{2.28}$$

$$B_0 \varphi = 0. \tag{2.29}$$

The particular solution is readily apparent:

$$\varphi(x,z) = -\int d^d x' \int_0^\infty dz' \sqrt{g'}\, \mathcal{D}_{\partial'}^2 G(x-x';z,z') \mathcal{Z}(x',z'), \tag{2.30}$$

where G is the AdS propagator, modulo factors of -1 depending on the convention employed. For the remainder of this section, we choose $\Delta = \Delta_-$. With this choice, the propagator is

$$G(x; z, z') = -\int \not{d}^d p \int_0^\infty dm\, (zz')^{\frac{d}{2}} J_{-\nu}(mz) J_{-\nu}(mz') \frac{m}{p^2 + m^2} e^{ip\cdot x}. \tag{2.31}$$

The homogenous solution, by construction, must be the same as the ϕ solution under the unmodified boundary conditions, and will thus be denoted ϕ_0. From this, it follows that

$$\phi(x, z) = \phi_0(x, z) + \int d^d x' \int_0^\infty dz' \sqrt{g'}\, \mathcal{D}^2_{z'} G^0(x - x'; z, z') \mathcal{Z}(x', z'). \tag{2.32}$$

To verify the procedure, let us consider a boundary source and determine the bulk-boundary propagator. Specifying $W = \epsilon^{d-\Delta+1}\phi_b\phi$ implies $\delta B = -\epsilon^{d-\Delta}\phi_b$. From Eqs.(2.26)&(2.27) we find, to leading order in ϵ, respectively,

$$\mathcal{Z} = a(x) z^{\Delta} + b(x) z^{d-\Delta} \tag{2.33}$$

$$b(x) = \frac{1}{2\nu}\phi_b(x). \tag{2.34}$$

The action of the integral kernel in Eq.(2.32) on z'^{Δ} is trivial and, consequently, only $b(x)$ matters. Since we are interested in the particular solution, we discard ϕ_0, leaving, in momentum space,

$$\phi(p, z) = \left[-\frac{1}{\Gamma(1-\nu)} \left(\frac{p}{2}\right)^{-\nu} z^{\frac{d}{2}} K_\nu(pz)\right] \phi_b(p). \tag{2.35}$$

As expected, the factor in brackets is indeed the momentum space incarnation of the bulk-boundary propagator,

$$\mathcal{K}(x, x'; z) = -\frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}} \frac{z^{\Delta_-}}{(z^2 + |x - x'|^2)^{\Delta_-}}. \tag{2.36}$$

Next, let us seek the modification to the bulk-boundary propagator with a double trace deformation:

$$\mathcal{L}_b = \epsilon^{d-\Delta+1}\phi_b\phi + \frac{1}{2}\lambda\epsilon^{d-2\Delta+1}\phi^2 \implies \delta B[\phi] = -\epsilon^{d-\Delta}\phi_b - \lambda\epsilon^{d-\Delta}\alpha. \tag{2.37}$$

This time, Eq.(2.27) implies

$$b(x) = \frac{1}{2\nu}\left[\phi_b(x) + \lambda\alpha(x)\right] \tag{2.38}$$

$$a(x) = 0, \tag{2.39}$$

and, consequently,

$$\phi(p,z) = -\left[\frac{1}{\Gamma(1-\nu)}\left(\frac{p}{2}\right)^{-\nu} z^{\frac{d}{2}} K_\nu(pz)\right]\left[\phi_b(p) + \lambda\alpha(p)\right]. \tag{2.40}$$

Since we are interested in the particular solution to the boundary source equation, α is not arbitrary and we must solve for it. Insisting Eq.(2.40) match Eq.(2.15) and solving for α results in:

$$\phi(p,z) = -\left[\left(\frac{1}{1+\frac{\Gamma(\nu)}{\Gamma(1-\nu)}\left(\frac{p}{2}\right)^{-2\nu}\frac{\lambda}{2}}\right)\frac{1}{\Gamma(1-\nu)}\left(\frac{p}{2}\right)^{-\nu} z^{\frac{d}{2}} K_\nu(pz)\right]\phi_b(p). \tag{2.41}$$

These results are so far known [35, 49], but careful attention should be paid to the procedure of throwing out ϕ_0 and, more precisely, solving for α. It is only with a priori knowledge that the particular solution gives us the bulk-boundary propagator that we have the luxury of demanding α arise only from the source ϕ_b. Indeed, had ϕ_0 not been discarded, it would scale to the boundary as

$$\phi_0(x,z) \underset{z\to 0}{\to} z^{-\Delta}A(x), \tag{2.42}$$

with $B(x) = 0$ ensured by the boundary counter-term.

Restricting to the $\mathcal{K}$ solution has its roots in demanding the bulk field be regular for $z \to \infty$ [4], but, since we are permitting the boundary fields to be dynamical, we should reconsider the origin of this restriction.

For general α and β, the field in the bulk takes the form

$$\phi(x,z) = \int d^d x' L_\alpha(x,x';z)\alpha(x') + 2\nu \int d^d x' L_\beta(x,x';z)\beta(x'), \tag{2.43}$$

where the lift kernels $L_{\alpha,\beta}$ are given by

$$L_\alpha(x,x';z) = \int đ^d p\, \Gamma(1-\nu) \left(\frac{p}{2}\right)^\nu z^{d/2} I_{-\nu}(pz) e^{ip\cdot(x-x')} \tag{2.44}$$

$$L_\beta(x,x';z) = \int đ^d p \left[\frac{1}{2}\Gamma(\nu) I_{-\nu}(pz) - \frac{1}{\Gamma(1-\nu)} K_\nu(pz)\right] \left(\frac{p}{2}\right)^{-\nu} z^{d/2} e^{ip\cdot(x-x')}. \tag{2.45}$$

Demanding $\beta = W'[\alpha]/2\nu$ customarily serve as a source for α requires

$$\alpha(x) = 2\nu \int d^d x'\, g(x-x') W'[\alpha], \tag{2.46}$$

where the *undeformed* boundary propagator is given by

$$g(x-x') = -\int đ^d p\, \frac{\Gamma(\nu)}{2\Gamma(1-\nu)} \left(\frac{p}{2}\right)^{-2\nu} e^{ip\cdot(x-x')} \propto \frac{1}{|\Delta x|^{2\Delta}}. \tag{2.47}$$

Inserting Eq.(2.46) into Eq.(2.43) yields

$$\phi(x,z) = 2\nu \int d^d x'\, \mathcal{K}(x-x';z) W'[\alpha], \tag{2.48}$$

which is precisely the form achieved by discarding ϕ_0 in the above formalism.

This indicates that including S_{ct} and S_W in the AdS action cannot be the entire story. An action for α, denoted herein as S_∂, that classically results in Eq.(2.46) must either be generated or appear by explicit insertion. Given the nature of the constraint, S_∂ should include terms that lead to a dynamical α, such as what would be contained in a generalized free theory in the large N limit. We argue in the following section that S_∂ is generated by integrating out the bulk.

As will be shown in the following section, multi-trace deformations require a proper quantum treatment, and so our chief classical analysis ends here; however, we consider multi-trace deformations and bulk wave functions and demonstrate that using the double-trace techniques of this section requires α to be classical in Appendix A.

2.4 Bulk and boundary partition functions redux

Conformal deformations generate interactions and, generally, quantum corrections. Indeed, quartic interactions anomalously break conformal invariance precisely due to the appearance of a renormalization scale arising from loop corrections. Since Witten diagrams will include vertices at the boundary, we should expect the bulk theory to inherit certain quantum effects. To demonstrate the appearance of such quantum corrections to bulk correlation functions at the level of the partition function, consider a multi-trace deformation constrained to a bulk UV brane at $z = \epsilon$ for a field ϕ dual to a conformal operator with scaling dimension Δ_-,

$$W[\phi] = \frac{1}{n}\lambda\epsilon^{-n\Delta}\phi^n(x,\epsilon) \tag{2.49}$$

$$\mathcal{L}_{ct} = \frac{1}{2}\epsilon\Delta_-\phi^2(x,\epsilon). \tag{2.50}$$

The generating partition function for bulk correlators is given by[2]

$$\begin{aligned} Z[J] = \int \mathcal{D}\phi \,\exp\Bigg[\Bigg(\int d^dx \int dz\,\sqrt{g}\,\frac{1}{2}\left\{z^2(\partial\phi)^2 - \left[\left(\frac{d}{2}\right)^2 - \nu^2\right]\phi^2\right\} + J\phi \\ + \int_{z=\epsilon} d^dx\,\epsilon^{-d-1}\left\{\frac{1}{2}\epsilon\Delta_-\phi^2 + \frac{1}{n}\lambda\epsilon^{d-n\Delta_-+1}\phi^n\right\}\Bigg) + S_\partial[\phi]\Bigg]. \end{aligned} \tag{2.51}$$

We can separate the quantum effects from the classical solutions by changing the integral measure:

$$\phi = \phi_{cl} + \theta \tag{2.52}$$

$$J = \sqrt{g}\left\{\nabla^2 + \left[\left(\frac{d}{2}\right)^2 - \nu^2\right]\right\}\phi_{cl}, \tag{2.53}$$

where, additionally, ϕ_{cl}'s boundary conditions are chosen to minimize the on-shell action,

$$\left[-\epsilon\partial_z\phi_{cl} + \Delta_-\phi_{cl} + \lambda\epsilon^{d-n\Delta_-}\phi_{cl}^{n-1}\right]\Big|_{z=\epsilon} = 0, \tag{2.54}$$

which leads to Eq.(2.16). The regular mode of the classical solution, α, is additionally forced by S_∂ to reproduce only the particular solution from Eq.(2.53)

[2]Here, the source is given its conventional symbol, J, which should not be mistaken for the Bessel functions appearing in the propagator.

Integrating the first term in Eq.(2.51) by parts and inserting Eqs.(2.53)&(2.54) into the result yields

$$Z[J] = \exp\left[\int d^dx \int dz\, \frac{1}{2} J\phi_{cl}[J]\right]$$
$$\times \int \mathcal{D}\theta\, \exp\left[S[\theta, J=0]\right] \exp\left[\int_\partial d^dx\, \frac{1}{n}\lambda\epsilon^{-n\Delta_-} \sum_{m=2}^{n-1} \begin{pmatrix} n \\ m \end{pmatrix} \theta^m \phi_{cl}^{n-m}[J] + S_\partial(\theta)\right], \tag{2.55}$$

where the functional dependence of ϕ_{cl} on J has been emphasized. Bulk correlators are specified by the integral kernels in the functional expansion of Eq.(2.55) in terms of J. Evidently, the coupling in the second line vanishes for $n \leq 2$ and quantum effects are manifestly absent; for $n > 2$, however, loop effects begin appearing, revealing the short coming of the usual approach taken for double trace deformations. For example, cubic interactions generate loop corrections to the bulk two point function at order λ^2. Explicitly,

$$\begin{aligned}
\langle\phi(x', z')\phi(x, z)\rangle &= \frac{\delta^2}{\delta J(x', z')\delta J(x, z)} Z[J]\Big|_{J=0} \\
&= \Bigg[\frac{\delta\phi_{cl}(x, z)}{\delta J(x', z')} + \lambda\epsilon^{-3\Delta_-} \int d^dy \langle\, \theta^2(y, \epsilon)\rangle \frac{\delta^2\phi_{cl}(y, \epsilon)}{\delta J(x', z')\delta J(x, z)} \\
&\quad + \lambda^2\epsilon^{-6\Delta_-} \int d^dy \int d^dy' \langle\theta^2(y', \epsilon)\theta^2(y, \epsilon)\rangle \frac{\phi_{cl}(y', \epsilon)}{\delta J(x', z')} \frac{\delta\phi_{cl}(y, \epsilon)}{\delta J(x, z)}\Bigg]\Bigg|_{J=0}.
\end{aligned} \tag{2.56}$$

The factors of $\epsilon^{-3\Delta_-}$ divide out the vanishing parts of the classical bulk correlators as they are taken to the boundary. The term naively proportional to λ vanishes since the vacuum specified by action S does not support vevs of θ or its composites; nonetheless, the functional derivatives of ϕ_{cl} are non-zero and arise from the second harmonic term of ϕ_{cl}, which itself is proportional to λ, ensuring the quantum corrections are indeed of order λ^2. Higher order corrections are implicitly contained in the usual loop term $\langle\theta^2(y')\theta^2(y)\rangle$; as addressed in Appendix A, the bulk-bulk and bulk-boundary propagators do not contribute larger λ corrections as they are non-vanishing only for non-vanishing J.

It follows that pulling correlators computed according to Eq.(2.55) to the boundary and computing correlators by evaluating the bulk partition function classically after pulling the source to the boundary are inequivalent. This is not to say the bulk and CFT partition functions are inequivalent, only that we should forgo computing the bulk fields classically. While the interior of the AdS action may be treated classically in the absence of bulk interactions, quantum effects appear through the boundary conditions. This suggests that the bulk theory should be understood as a lift of a quantum boundary theory. To clarify, this simply means the bulk is a classical boundary-value problem with quantum dynamics governing the behavior of the boundary conditions. To formalize this notion, we wish to express the generating bulk partition function as a functional of the boundary fields. We follow the procedure of dividing the generating path integral into IR ($\epsilon + \delta < z$) and UV ($\epsilon \leq z \leq \epsilon + \delta$)

contributions [11, 50],

$$Z[J] = \int_{IR} \mathcal{D}\phi \, \exp\left[S_{bulk}[\phi] + \int d^d x \int_{\epsilon+\delta}^{\infty} dz\, J\phi\right] \times \int_{UV} \mathcal{D}\phi \, \exp\left[S[\phi] + \int d^d x \int_{\epsilon}^{\epsilon+\delta} dz\, J\phi\right], \tag{2.57}$$

and let $\delta \to 0$. S and S_{bulk} are given by Eq.(2.17) and we will choose $\Delta = \Delta_-$ as the scaling dimension of the undeformed boundary theory. As before, we separate the classical and quantum effects, but break down the quantum corrections in the IR differently:

$$\phi(x, z) = \phi_0(x, z) + \phi_{cl}(x, z) + \theta(x, z), \tag{2.58}$$

$$\phi_0(x, z) = \frac{1}{2\nu}\epsilon^{-\Delta_-} \int d^d x' L_\alpha(x, x'; z) \left[\Delta_+ \theta(x, \epsilon) - \epsilon \partial_z \theta(x, \epsilon)\right] - \epsilon^{-\Delta_+} \int d^d x' L_\beta(x, x'; z) \left[\Delta_- \epsilon \theta(x, \epsilon) - \epsilon \partial_z \theta(x, \epsilon)\right], \tag{2.59}$$

$$\phi_{cl}(x, z) = \int d^d x' \int dz'\, G(x - x'; z, z') J(x', z') \tag{2.60}$$

The ϕ_0 term contained in the IR bulk field is, explicitly, a lift of the homogeneous boundary (UV) fields. As before, θ is the quantum perturbation. With this ansatz, the generating partition function becomes

$$Z[J] \underset{\delta\to 0}{=} \exp\left[\int d^d x \int dz\, \frac{1}{2} J\phi_{cl}\right] \int_{IR} \mathcal{D}\theta \exp\left[S_{bulk}[\theta]\right] \times \int \mathcal{D}\theta(\epsilon+\delta)\mathcal{D}\theta(\epsilon) \Bigg[\int_{z=\epsilon} d^d x \left(-\frac{1}{2}\epsilon^{-d+1}\phi_0 \partial_z \phi_0 + \frac{1}{2}\epsilon^{-d}\Delta_- \phi_0^2\right) + S_W[\phi_{cl} + \phi_0] \Bigg]. \tag{2.61}$$

The path integral variable $\theta(\epsilon+\delta)$ appears only through the derivative $\partial_z\phi_0 = \frac{\theta(\epsilon+\delta)-\theta(\epsilon)}{\delta}$. It is more convenient to work in terms of the usual functions α and β that parameterize the asymptotic behavior of ϕ_0. Writing

$$\begin{pmatrix} \phi_0(\epsilon) \\ \\ \phi_0(\epsilon+\delta) \end{pmatrix} = \begin{pmatrix} \epsilon^{\Delta_-} & \epsilon^{\Delta_+} \\ \\ (\epsilon+\delta)^{\Delta_-} & (\epsilon+\delta)^{\Delta_+} \end{pmatrix} \begin{pmatrix} \alpha \\ \\ \beta \end{pmatrix}, \tag{2.62}$$

the generating partition function becomes

$$\begin{aligned} Z[J] =& \exp\left[\int d^dx \int dz\, \frac{1}{2} J\phi_{cl}\right] \int_{IR} \mathcal{D}\theta \exp\left[S_{bulk}[\theta]\right] \\ &\times \int \mathcal{D}\alpha\mathcal{D}\beta \exp\left[\int d^dx\, \left(-\nu\alpha\beta - \nu\epsilon^{2\nu}\beta^2 + W[\alpha + \epsilon^{2\nu}\beta + \epsilon^{-\Delta_-}\phi_{cl}]\right)\right], \end{aligned} \tag{2.63}$$

modulo irrelevant factors of δ and ϵ that arise from the Jacobian from our change of integral measure.

In the absence of a bulk source and as $\epsilon \to 0$, we expect to recover the correct CFT partition function with the appropriate S_∂. The derivative terms in the IR partition function may be integrated by parts, leaving only boundary terms, and the bulk fields can be integrated out resulting in

$$\begin{aligned} &\exp\left[\frac{1}{2}\int d^dx dz \sqrt{g} \int d^dx' dz' \sqrt{g'} G(x-x';z,z') \frac{\delta}{\delta\theta(x,z)} \frac{\delta}{\delta\theta(x',z')}\right] \\ &\times \exp\left[-\int_{z=\epsilon} d^dy \epsilon^{-d+1} \theta\partial_z\theta\right]. \end{aligned} \tag{2.64}$$

We do not offer a rigorous proof that this generates the necessary terms, but point out that for the AdS/CFT dictionary to hold in the absence of deformations and for the counter terms currently used in the literature to be correct, Eq.(2.64) must evaluate to[3]

$$\exp\left[\int d^d x \left(\nu\alpha\beta + S_\partial[\alpha]\right)\right]. \tag{2.65}$$

Inserting this into Eq.(2.63) finally yields

$$\begin{aligned} Z_{AdS}[J] = &\exp\left[\int d^d x \int dz\, \frac{1}{2} J\phi_{cl}\right] \\ &\times \int \mathcal{D}\alpha\mathcal{D}\beta \exp\left[\int d^d x\, \left(-\nu\epsilon^{2\nu}\beta^2 + W[\alpha + \epsilon^{2\nu}\beta + \epsilon^{-\Delta_-}\phi_{cl}]\right) + S_\partial[\alpha]\right]. \end{aligned} \tag{2.66}$$

Evidently, β is an auxiliary field. Integrating it out sets, to leading order in ϵ,

$$\beta = \frac{1}{2\nu} W'[\alpha], \tag{2.67}$$

thus recovering the usual boundary conditions.

Meanwhile, the functional integral over α and the written dependence of S_∂ on α are formalities that simply instruct us to evaluate α as $\mathcal{O}$ (up to normalization factors) given the appropriate boundary CFT.

It is immediately apparent from Eq.(2.66) that

$$\lim_{\epsilon\to 0} Z_{AdS}[\phi_b] = Z_{CFT}[\phi_b]. \tag{2.68}$$

[3]The S_∂ term should be expected from the AdS/CFT story; the $\nu\alpha\beta$ term is necessary to counter the $-\nu\alpha\beta$ term one would obtain by evaluating the bulk action on-shell and integrating by parts. Without canceling it out, the boundary correlators would not be evaluated as expected in the dual CFT.

2.5 Bulk correlation functions and the lift formalism

With Eq.(2.66) establishing an appropriate dictionary for multi-trace deformations, we may compute bulk correlators and compare them to expectations from Witten diagrams.

The expression of the bulk partition function in terms of a boundary partition function through the construction of the bulk fields from boundary fields in the previous section encourages the literal interpretation of the bulk theory as a theory of quantum boundary conditions. This leads us to consider the use of smearing functions to compute bulk correlators from boundary correlators. We review the use of smearing functions, and go on to develop an alternative, but related, formalism.

2.5.1 Correlating

It immediately follows from Eq.(2.66) that bulk correlation functions take the form

$$\begin{aligned}
&\langle \phi(x_1, z_1)\phi(x_2, z_2)\dots\phi(x_{n-1}, z_{n-1})\phi(x_n, z_n)\rangle = \\
&G(x_2 - x_1; z_2, z_1)\dots G(x_n - x_{n-1}; z_n, z_{n-1}) + \{perm\} \\
&+\prod_{i=1}^{n}\left[\int d^d y_i\, \mathcal{K}(x_i - y_i; z_i)\right]\Gamma_n(y_1, y_2, \dots, y_n),
\end{aligned} \tag{2.69}$$

where the vertex function, Γ_n, arises from derivatives of W. This is in agreement with the form that follows from Witten diagrams.

Computed via diagrams, the two-point function, shown in Fig. 2.1, for instance, is given in momentum space by

$$\langle \phi(-p, z_1)\phi(p, z_2)\rangle = G(p; z_1, z_2) + \mathcal{K}(p, z_1)\frac{\Sigma(p)}{1 - g(p)\Sigma(p)}\mathcal{K}(p, z_2), \tag{2.70}$$

where $\Sigma(p)$ is the usual sum of 1PI diagrams at the boundary. Using Eq.(2.66), we find, schematically,

$$\Gamma_2 = \langle W'[\alpha]W'[\alpha] + W''[\alpha]\rangle. \tag{2.71}$$

Identifying this with Eq.(2.70) requires

$$\frac{\Sigma}{1 - g\Sigma} = \langle W'[\alpha]W'[\alpha] + W''[\alpha]\rangle. \tag{2.72}$$

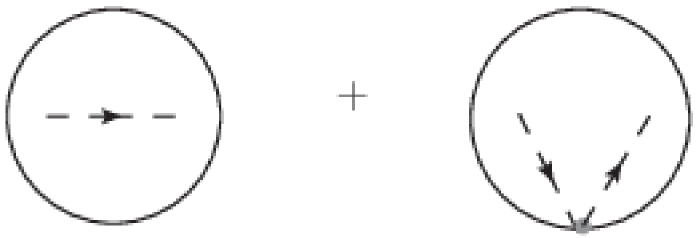

Figure 2.1: Witten diagram for the bulk two-point function. The sum over 1PI parts at the boundary is absorbed into the vertex.

For a double-trace deformation, $W = \frac{1}{2}\lambda\alpha^2$, we find

$$\langle W'[\alpha]W'[\alpha] + W''[\alpha]\rangle = \frac{\lambda}{1 - g(p)\lambda}, \tag{2.73}$$

where the boundary correlators are computed according to the usual CFT rules, including summing over all double trace insertions. This predicts $\Sigma = \lambda$, as would be expected diagrammatically from adding a mass term.

From the cubic deformation $W[\alpha] = \frac{1}{6}\lambda\alpha^3$, we find

$$\langle W'[\alpha](y_1)W'[\alpha](y_2) = \left(\frac{1}{2}\lambda\right)^2 \langle \alpha^2(y_1)\alpha^2(y_2)\rangle. \tag{2.74}$$

The boundary correlator, which is represented diagrammatically as the bracketed factor in Fig. 2.2, evaluates to[4]

$$\left(\frac{1}{2}\lambda\right)^2 \langle \alpha^2(-p)\alpha^2(p)\rangle = \frac{\Sigma(p)}{1-g(p)\Sigma(p)}, \tag{2.75}$$

with the usual cubic 1PI diagram for Σ,

$$\Sigma(p) \propto \lambda^2 \int đ^d l \, l^{-2\nu}(p-l)^{-2\nu} + \ldots, \tag{2.76}$$

manifestly agreeing with the Witten diagram.

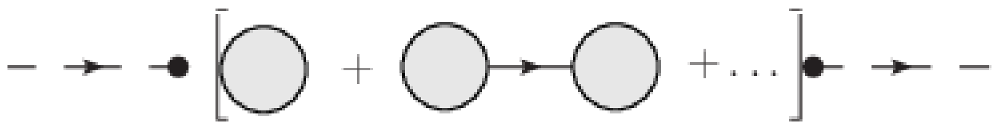

Figure 2.2: A diagrammatic representation of the vertex function $\langle \alpha^2\alpha^2\rangle$. The 1PI diagrams in brackets are indeed amputated.

The bulk three-point function arising from the cubic deformation is similarly easy

[4]A factor of 2 appears at each vertex due to symmetry.

to assess using Eq.(2.66):

$$\begin{aligned}\langle\phi(x_1,z_1)\phi(x_2,z_2)\phi(x_3,z_3)\rangle = &\prod_{i=1}^{3}\left[\int d^d y_i\,\mathcal{K}(x_i-y_i;z_1)\right]\\ &\times\left[\lambda\int d^d y\,\delta^d(y_1-y)\delta^d(y_2-y)\delta^d(y_3-y)\right.\\ &\left.+\left(\frac{1}{2}\lambda\right)^3\langle\alpha^2(y_1)\alpha^2(y_2)\alpha^2(y_3)\rangle\right]. \end{aligned} \tag{2.77}$$

As should be expected, loop effects enter through the boundary correlator $\langle\alpha^2(y_1)\alpha^2(y_2)\alpha^2(y_3)\rangle$.

The agreement between bulk correlators computed via diagrams and those computed using Eq.(2.66) confirms Eq.(2.66) as the appropriate AdS/CFT partition function to compute bulk correlators with boundary deformations.

To this end, given the construction of the bulk ϕ in the previous section as a lift of the boundary fields, we should equivalently be able to compute bulk correlators by using Eq.(2.43) and computing the resulting correlators of α and β using the boundary path integral in Eq.(2.66). This is reminiscent of the effort to construct bulk observables from boundary operators using smearing functions. Before developing our lift formalism, it is worthwhile to first very briefly review the program and status of smearing functions.

2.5.2 Smearing

The goal of the smearing program is to construct bulk operators from their CFT duals through a *linear* integral operation:

$$\phi(B) = \int dbK(B;b)\mathcal{O}(b), \tag{2.78}$$

where the integral kernel $K(B;b)$ is the smearing function that integrates over a boundary coordinate b to generate a field at bulk coordinate B.

Without interactions, $K(B;b)$ was found in global coordinates in [37] through both a Green function approach and mode function expansion. A review of the mode expansion approach is give in [41] and is sketched here.

A free bulk field may be generically expanded as

$$\phi(B) = \sum_n f_n(B)a_n + h.c. \tag{2.79}$$

where f_n denotes the wave function (eigenfunction) with quantum numbers (eigenvalues) n satisfying the classical bulk equations of motion, and a_n ($a_n^\dagger$) is the associated annihilation (creation) operator. With an appropriate normalization, $\{f_n\}$ forms an orthonormal set and a_n consequently satisfies the appropriate algebra, $[a_n, a_m^\dagger] = \delta_{nm}$.

Carrying $\phi(B)$ to the boundary, $B \to b$, maps to $\mathcal{O}(b)$ in the usual way, implying $\mathcal{O}$ then inherits a related mode expansion:

$$\mathcal{O}(b) = \sum_n f_{\partial,n}(b)a_n + h.c. \tag{2.80}$$

Provided an appropriate foliation of AdS, f_n can be defined to be orthonormal along radial slices on AdS and thus remain orthonormal at the boundary, implying $\langle f_{\partial,n}|f^{\dagger}_{\partial,m}\rangle = \delta_{nm}$. With this, Eq.(2.79) becomes

$$\phi(B) = \sum_n f_n(B)\langle f^{\dagger}_{\partial,n}\mathcal{O}\rangle + h.c. \tag{2.81}$$

From this, the schematic form of the smearing function can be immediately extracted:

$$K(B;b) = \sum_n f_n(B) f^{\dagger}_{\partial,n}(b) + h.c. \tag{2.82}$$

From Eq.(2.82), it immediately follows that $K(B;b)$ satisfies the classical equations of motion in the bulk.

The existence of $K(B;b)$ is predicated on the convergence and support of Eq.(2.82) for nontrivial B. For certain backgrounds, such as in the presence of a black hole, the sum is non-convergent or lacks support [41], and certain constructions result in a non-causal map ($\lim_{B\to b} K(B;b') \neq \delta(b-b')$) [42]. Nonetheless, smearing provides a powerful means of probing conformal theories dual to free AdS theories. We aim to develop an alternative construction of bulk fields in the same spirit as smearing.

2.5.3 Lifting

Instead of seeking a linear operation that maps $\mathcal{O}$ to ϕ, we employ Eq.(2.43) with a caveat on the form of β to be lifted that will be derived here. To develop this formalism, consider first the undeformed bulk-boundary two point function:

$$\epsilon^{-\Delta_-}\langle\phi(y,\epsilon)\phi(x,z)\rangle = \int d^dx'\ [L_\alpha(x,x';z)\langle\alpha(y)\alpha(x')\rangle + 2\nu L_\beta(x,x';z)\langle\alpha(y)\beta(x')\rangle]\,. \tag{2.83}$$

The first boundary correlator is $\langle\alpha(y)\alpha(x')\rangle = g(y-x')$ (the boundary propagator); the second correlator must, perhaps surprisingly, evaluate to a local term, $\langle\alpha(y)\beta(x')\rangle = \frac{1}{2\nu}\delta^d(y-x')$, to obtain the correct bulk-boundary propagator. However, integrating β out in the absence of a deformation in Eq.(2.66) sets $\beta = 0$. This suggests that, at the level of operators, we must make the substitution

$$\beta \to \beta + \beta_0 \tag{2.84}$$

in Eq.(2.43). The action of this undeformed $\beta(x)$ on $f[\alpha]$ can be viewed as the functional derivative when evaluating correlators:

$$\beta_0(x) = \frac{1}{2\nu}\frac{\delta}{\delta\alpha(x)}. \tag{2.85}$$

Next, we evaluate the undeformed bulk-bulk two-point function using the same technique to find

$$\begin{aligned}\langle\phi(-p,z)\phi(p,z')\rangle = &-(zz')^{d/2}\left[I_{-\nu}(pz)K_\nu(pz') + I_{-\nu}(pz')K_\nu(pz) + I_{-\nu}(pz)I_{-\nu}(pz')\right]\\ &+ (2\nu)^2 L_\beta.\langle(\beta+\beta_0)(\beta+\beta_0)\rangle.L_\beta,\end{aligned} \tag{2.86}$$

where the '.' binary operator denotes the integration over the common boundary coordinates of the adjacent objects. Since there are no interactions, the second line vanishes. The first of the three remaining terms is the two-point function for $z' > z$, the second is for $z > z'$, and the third is for when $z = z'$. We have encountered a problem with treating the bulk as the lift of a CFT: the boundary has no knowledge of the relative $z-$positions of our correlators in the bulk. This issue can be fixed by inserting a $z-$ordering operator, Z, into correlators: $\langle\phi\phi\rangle \rightarrow \langle Z\phi\phi\rangle$. The operator annihilates β_0 for the field ϕ with the *smaller* z in Wick-contracted $\phi\phi$ pairs. It should be noted that the operator only affects the non-interacting Witten diagrams as the diagrams containing boundary interactions are tautologically ordered according to the Z prescription.

With boundary deformations turned on, β takes its usual form:

$$\beta = \frac{1}{2\nu} W'[\alpha]. \tag{2.87}$$

With this feature, we find

$$\begin{aligned}\langle Z\phi(x,z)\phi(x',z')\rangle =& G(x-x';z,z') + L_\alpha.\langle\alpha\alpha - \alpha_0\alpha_0\rangle.L_\alpha \\ &+ L_\alpha.\langle\alpha W'[\alpha]\rangle.L_\beta + L_\beta.\langle W'[\alpha]\alpha\rangle.L_\alpha \\ &+ L_\beta.\langle W'[\alpha](y)W'[\alpha](y') + W''[\alpha]\delta^d(y-y')\rangle.L_\beta,\end{aligned} \tag{2.88}$$

where α_0 is the undeformed α. It follows from pulling Eq.(2.70) to the boundary, identifying the near-boundary bulk-boundary two-point function with

$$\epsilon^{-\Delta_-}\langle\alpha(x)\phi(x',\epsilon)\rangle \approx \langle\alpha(x)(\alpha(x') + \beta(x')\epsilon^{2\nu})\rangle, \tag{2.89}$$

and demanding consistency among already confirmed results that, in momentum space,

$$\langle \alpha W'[\alpha] \rangle = \frac{g\Sigma}{1 - g\Sigma}. \tag{2.90}$$

With this identification, Eq.(2.88) reproduces Eq.(2.70):

$$\begin{aligned} \langle Z\phi(x,z)\phi(x',z') \rangle =& G(x-x';z,z') \\ &+ \mathcal{K}(x-y;z).\langle W'[\alpha](y)W'[\alpha](y') \\ &+ W''[\alpha]\delta^d(y-y') \rangle.\mathcal{K}(y'-x';z'). \end{aligned} \tag{2.91}$$

All rules for computing bulk correlators from boundary data are now in place, completing the formalism.

It is worthwhile noting the subtle distinctions between lifting and smearing. While the lift and smear kernels both satisfy the linear classical bulk equations of motion in the absence of bulk interactions, lifting generally provides a nonlinear map from CFT operators to bulk fields in the presence of boundary interactions as a consequence of the boundary conditions and an affine map[5] in the absence of deformations. The mapping also provides a means of reproducing the results of Witten diagrams at the level of correlators without the need to adjust a tower of coefficients to cancel noncausal effects in the presence of (boundary) interactions [39]. The lift kernel also manifestly approaches a delta function when taken to the boundary, ensuring bulk fields constructed from boundary operators map correctly when taken to the boundary.

[5] A linear map is one of the form $y = mx$, while an affine map takes the form $y = mx + b$.

It is also worthwhile to recapitulate the results of this and the last few sections:

- The computation of bulk correlators in the presence of multi-trace boundary deformations should be carried out with Eq.(2.66) as the generating partition function.

- Pulling bulk correlators to the boundary in the usual way returns the same results as computing CFT correlators with conformal deformations. That is to say, $Z_{CFT}[\phi_b] = \lim_{\epsilon \to 0} Z_{AdS}[\phi_b, \epsilon]$.

- Bulk observables can be constructed from CFT observables via the lift provided by Eq.(2.43). The formalism makes the identification $\beta = \frac{1}{2\nu}\left[W'[\alpha] + \frac{\delta}{\delta\alpha}\right]$ when computing correlators, and the operator Z was introduced to order the lift operation by z-coordinate.

- There is a strong connection between the β term in the bulk and the 1PI diagram when computing correlators. In particular, the results Eq.(2.72) and Eq.(2.90) are interesting and useful.

2.6 Dilatation spectrum and RG flow

We now wish to use the results of the lift formalism to find a generic form for the conformal dimension of a dual operator $\mathcal{O}$ as a function of energy scale in the presence of multi-trace deformations. The results are the first steps to the multi-trace generalization of [51].

In the absence of deformations, the dilatation spectrum is dual to the bulk mass spectrum via the mapping $m^2 = \Delta(\Delta - d)$. The inclusion of conformal deformations triggers RG flow such that the IR spectrum can often be extracted from the UV spectrum. For instance, in the presence of double-trace deformations, operators with dimension $\Delta_-(= \frac{d}{2} - \nu \leq \frac{d}{2})$ in the UV flow to a $\Delta_+(= \frac{d}{2} + \nu)$ fixed point in the IR [49]. The conformal dominance program exploits the UV conformal basis to construct IR mass states for certain deformations [30, 31]. In what follows we restrict the scaling dimension to $\Delta = \Delta_-$. Many results can be immediately extended to $\Delta = \Delta_+$, however we are chiefly concerned with the RG flow of the CFTs between potential fixed points.

Dilatation eigenstates in the undeformed CFT are created by placing an operator at the origin: $|0\rangle = \mathcal{O}(0)|\Omega\rangle$. It follows from the identity $[D, \mathcal{O}(x)] = (x \cdot \partial + \Delta_-)\,\mathcal{O}(x)$ that

$$\langle x|D|0\rangle = \Delta_- \langle x|0\rangle = -\,(x \cdot \partial + \Delta_-)\,\langle x|0\rangle. \tag{2.92}$$

Evidently, the CFT two-point functions are eigenfunctions of the differential representation of the dilatation operator.

When lifting to the bulk, the scaling dimension is replaced by differentiation with respect to z: $(x\cdot\partial+\Delta_-)\mathcal{O} \to (x\cdot\partial+z\partial_z)\phi$. With this replacement, the bulk-boundary propagator is found to be an eigenfunction of the dilatation operator:

$$\langle\phi(x,z)|D|0\rangle = -\left(x\cdot\partial + z\partial_z\right)\mathcal{K}(x;z) = \Delta_-\mathcal{K}(x;z). \tag{2.93}$$

Even more readily, and perhaps crucially, the classical field ϕ is an eigenfunction of $z\partial_z$ as $z \to 0$ with eigenvalue Δ_-.

When deformations are introduced, the story becomes more complex. In momentum space, the boundary two-point function remains an eigenfunction of the dilatation operator, but the eigenvalue gains a momentum-dependence:

$$\left(p\cdot\frac{\partial}{\partial p} + \Delta_-\right)\langle p|0\rangle = \left[\Delta_- + \frac{g(p)}{1-g(p)\Sigma(p)}\left(p\cdot\frac{\partial}{\partial p} - 2\nu\right)\Sigma(p)\right]\langle p|0\rangle. \tag{2.94}$$

As expected for a double trace deformation ($\Sigma = const.$) in the UV, we find $g\Sigma \ll 1$, which leads to $\Delta = \Delta_-$; in the IR, we find$g\Sigma \gg 1$, which leads to $\Delta = \Delta_+$.

The bulk-boundary two point function also remains an eigenfunction of the dilatation operator with the same eigenvalue as its boundary counterpart:

$$\begin{aligned}&\left(p\cdot\frac{\partial}{\partial p} + z\partial_z\right)\frac{1}{1-g(p)\Sigma(p)}\mathcal{K}(p;z) = \\ &\left[\Delta_- + \frac{g(p)}{1-g(p)\Sigma(p)}\left(p\cdot\frac{\partial}{\partial p} - 2\nu\right)\Sigma(p)\right]\frac{1}{1-g(p)\Sigma(p)}\mathcal{K}(p;z).\end{aligned} \tag{2.95}$$

The relation between the bulk operator $z\partial_z$ and the conformal dimension becomes more obscure in the presence of deformations. We could attempt to demand the field ϕ remain the eigenfunction of the operator at the boundary designated by the $z = \epsilon$ cutoff and identify the eigenvalue with Δ. Carrying through with the procedure with a double-trace deformation and keeping next to leading order terms in ϵ for ϕ results in

$$\Delta = \phi^{-1}\epsilon\partial_\epsilon\phi = \Delta_- + \frac{\lambda\epsilon^{2\nu}}{1+\frac{\lambda}{2\nu}\epsilon^{2\nu}}. \tag{2.96}$$

If we interpret the UV brane on which the bulk theory terminates as the inverse of the renormalization scale, $\epsilon \propto \mu^{-1}$, the RG flow in Eq.(2.96) matches exactly Eq.(2.94) for double trace deformations. The procedure as presented is serendipitous for double-trace deformations since β is classically linear in α, allowing the field to completely divide out; this does not occur for more complicated deformations. However, the promising connection between the z-direction in the bulk and RG flow at the boundary begs for the procedure to be generalized.

When transitioning to the quantum formulation, we should expect to deal with correlators of the fields instead of the fields themselves. Classically, we may multiply and divide Eq.(2.96) by α, so transitioning suggests we compute Δ in momentum space by pulling the bulk-boundary propagator to the UV cutoff and evaluating the momentum at this UV scale:

$$\Delta \underset{\epsilon\to 0}{=} \left.\frac{\epsilon\partial_\epsilon\left[\langle\phi\alpha\rangle(p,\epsilon)\right]}{\left[\langle\phi\alpha\rangle(p,\epsilon)\right]}\right|_{|p|=\epsilon^{-1}} \tag{2.97}$$

for $\epsilon \to 0$. This procedure actually trivially follows from Eq.(2.95) by simply demanding the momentum be evaluated at the UV cutoff. Physically, Eq.(2.97) says that the scaling dimension is a measure of how the bulk-boundary propagator changes as the location of the UV brane is shifted while keeping the energy near the renormalization scale.

Using Eqs.(2.43)&(2.67) to expand the bulk fields in Eq.(2.97) in terms of α, inserting Eq.(2.90) into the result, and setting $\epsilon \to \mu^{-1}$ yields

$$\Delta = \Delta_- + \frac{\frac{\mu^{-2\nu}}{2\nu}\left[2\nu - \mu\partial_\mu\right]\Sigma(\mu)}{1 + \frac{\mu^{-2\nu}}{2\nu}\Sigma(\mu)}. \tag{2.98}$$

Once again, the correct RG flow has been recovered. This indicates that the running of a deformed CFT from the UV to the IR may be studied in AdS by ending the theory on a brane at $z = \epsilon$ and computing appropriate quantities by setting the renormalization scale $\mu \to \epsilon^{-1}$. This procedure is in independent agreement with the holographic RG program of interpreting the classical evolution of fields in the radial direction in AdS as RG flow at the boundary. The dilatation spectrum can be explicitly computed in the bulk using Eq.(2.97), and we wish to emphasize the necessity of the β_0 piece defined via the lift formalism in computing the bulk correlators. It is not difficult to imagine a similar analytical procedure should hold for the mass-squared spectrum, although the details are not immediately apparent.

2.7 Discussion

We have constructed rules utilizing modified boundary conditions in AdS to compute bulk correlators through the use of an appropriate AdS/CFT partition function and via a lift formalism for theories subject to a conformal deformation. The partition function explicitly relates the AdS theory to the CFT theory and establishes the AdS side in the absence of bulk interactions as a theory that evolves classically in the z- (radial-) direction with boundary conditions subject to quantum effects. The lift formalism, as an alternative to smearing, provides the inverse of the usual boundary-scaling dictionary that relates the bulk and boundary operators ($\mathcal{O} = \lim_{z\to 0} z^{-\Delta}\phi(z)$). Utilizing the results of the lift formalism, a formula to compute the conformal dimension of CFT operators as a function of energy scale using AdS correlators was derived.

We have not considered the obstructions that may hinder obtaining an appropriate smearing kernel. While we expect the lift formalism to fail or require modification when bulk interactions are turned on as the bulk would no longer evolve classically in the z-direction, it is our hope that it remains valid for asymptotically AdS spaces so that semiclassical gravity may be studied.

For now, only the running of the scaling dimension with the renormalization scale was considered since its fundamental role in the AdS/CFT correspondence makes it easy to handle. A similar strategy of finding an appropriate differential operator in the bulk and writing down a ratio of correlators may likely be employed to compute the dependence of mass-squared elements on the renormalization scale to approach conformal dominance from a bulk perspective. It would be of interest to explore this approach in the context of the c-theorem.

Chapter 3

Applying the Formalism: the Goldstone Equivalence Theorem

3.1 Introduction

The Goldstone equivalence theorem (hereafter 'ET') relates the S-matrices of processes involving longitudinally polarized massive gauge bosons to the those of scalars when the scattering energies are large compared to the gauge boson's mass [52]. A diagrammatic depiction of the ET is shown in Fig. 3.1, and the algebraic form is given formally by

$$S[A_L] \underset{m_A^2/s\to 0}{=} (-i)^n S[\pi] \tag{3.1}$$

for n replacements of gauge bosons to scalars, $A_L \to \pi$. Here, m_A denotes the boson's mass, $\sqrt{s}$ is the center of mass energy, and S is the S-matrix element for some process.

The high-energy limit $m_A^2/s \to 0$ corresponds to the massless limit of a spontaneously broken gauge theory, when the Goldstone mode is decoupled.

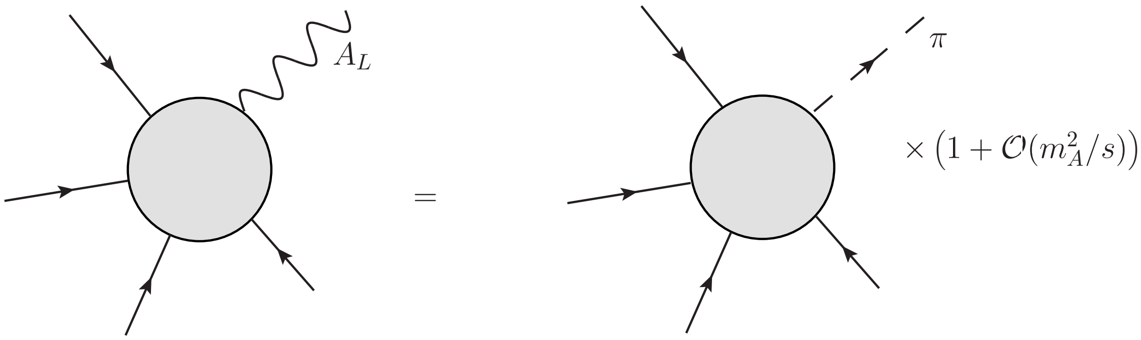

Figure 3.1: Diagrammatic representation of the Goldstone equivalence theorem. Here we represent the gauge boson as W in its longitudinally polarized state and its associated Goldstone mode ϕ. Figure has been adapted from [53].

A somewhat more sophisticated way of phrasing the ET is to note that the Ward identity for an amplitude involving massive spin-1 gauge fields can be decomposed into a separate gauge field and a scalar mode — these correspond to the 1PI vertex function and the scalar mode created by a current in a spontaneously broken gauge theory, respectively[1]. The ET then implies that the effect of contracting the vertex function with the longitudinal polarization vector reproduces this Ward identity at high energies. This line of analysis can be examined for theories of higher spin massive fields (see, for example, [55]) and the ET is a strong statement about the limiting behavior of scattering amplitudes in these theories. One need not delve far into the literature to find other uses of the ET. For example, one may go to the unitary

[1]This assumes the current is only between on-shell particles. A more careful statement of the ET can be derived for more complicated processes (see, e.g. [54])

gauge to absorb the perturbations of the inflationary scalar field into the graviton component g_{00} [56]. By the ET, it is possible to write down the effective action that corresponds to the scalar piece that is an excellent approximation at energies much greater than $\sqrt{-\dot{H}}$, taking H to be the canonical Hubble scale.

This suggests that the most immediate consequence of Goldstone equivalency is that in the appropriate regime of validity, one may calculate scalar correlation functions instead of correlation functions which involve gauge fields. Such an application can be very useful, given that gauge field correlation functions typically contain index structures that can render calculations prohibitively difficult. Of course, more indirect and subtle consequences fall out of the ET. For example, the ET implies a nontrivial cancellation of tree level Feynman diagrams involving massive gauge fields of the Standard Model so as to preserve unitarity [53], although this will not be our central focus.

In flat space, the Ward identity for massive gauge bosons and, consequently, the ET are rather straightforward to show. The Schwinger-Dyson equations provide something akin to a position space statement of the Ward identity by relating the position space vertex functions (amputated Green functions) for processes involving longitudinally polarized gauge bosons to their associated Goldstone bosons (Γ^{μ}_{A} and Γ_{π}, respectively):

$$\partial_{\mu}\Gamma^{\mu}_{A}(x) \propto \Gamma_{\pi}(x). \tag{3.2}$$

The LSZ formula then provides a map from position space correlation functions to

S-matrix elements. The longitudinal polarization vector for large scattering energies is known to be approximately the momentum of the particle: $\epsilon_L(p) \approx m_A^{-1} p_\mu + \mathcal{O}\left(\frac{m_A^2}{E^2}\right)$. Therefore, one can replace the polarization vector of the incoming vector particle in an S-matrix calculation with its momentum. Pulling the momentum into the LSZ integral and noting that the wave functions appearing in the formula are of the form $e^{ip\cdot x}$, the momentum can be recast in position space as a derivative and the typical Ward identity is found via the Schwinger-Dyson relations:

$$p_\mu \Gamma^\mu_A(p) \sim \int d^{d+1}x\, e^{ip\cdot x} \partial_\mu \Gamma^\mu_A(x) \sim \int d^{d+1}x\, e^{ip\cdot x} \Gamma_\pi(x) = \Gamma_\pi(p). \tag{3.3}$$

Thus, vector legs with large momentum can be replaced by their associated Goldstone boson. In the center of mass frame, large s implies all external legs have large momentum, and all vector legs can then be replaced with scalar legs. Since the result is a Lorentz invariant statement, the ET is recovered in all frames.

Since the ET applies at large scattering energies, we trivially expect a similar proof to hold in AdS as one may obtain the flat space limit by sending the bulk curvature scale to zero, which precisely corresponds to bulk interactions where center of mass energies are "large". However, our goal is less trivial as we will derive a sharp statement of the ET through the direct analysis of (1) massive AdS gauge fields and (2) dual CFT currents.

The first is achieved by generalizing the usual AdS/CFT dictionary [5, 57–59]. Our goal is to obtain a similar proof of the ET by extracting descendant vector states and then comparing the result to primary scalar states via the LSZ formula and

Schwinger-Dyson relations as above. However, there are several ambiguities that one must address first. For instance, LSZ and Schwinger-Dyson imply an individual vector with large momentum can be replaced by a scalar. The Lorentzian generalization of this statement is the ET and relies on the fact that, at least in the center of mass frame, large s implies large $\vec{p}$. In a tree level diagram, this means that large exchanged masses imply large s. In AdS, what is the analog to large $\vec{p}$, and what do large exchanged masses in AdS require of the quantum numbers of the external states for the exchange to occur?

At leading order when a heavy particle is exchanged, the crux of the result can be stated in momentum space, in analogy to Eq. (3.1), as

$$S[A_z] = \left[-\sqrt{(\Delta_J - (d-1))(\Delta_J - 1)}\right]^N \left(\prod_{i=1}^{N} |p_i|^{-1}\right) S[\pi] \tag{3.4}$$

when

$$\frac{\Delta_J}{\Delta} \ll 1, \tag{3.5}$$

where Δ_J is the scaling dimension of the gauge bosons, and Δ is the scaling dimension of the exchanged particle. The functions $S[O]$ are defined by

$$S[O] = \left(\prod_{i=1}^{N} \int d^d x_i \int_0^\infty dz_i \sqrt{g(z_i)}\, f^{(i)}(p_i; x_i, z_i)\right) \cdot \Gamma_O(\{x_i, z_i\}), \tag{3.6}$$

where $f^{(i)}(p_i)$ is the mode function associated with the ith external particle and Γ_O is the vertex function mediating the interaction. Generally, f and Γ can have index structure. Under the AdS/CFT dictionary, $S[O]$ may be interpreted as the Fourier transform of the conformal correlators of the operators dual to the bulk fields.

The analogue to the large energy limit for a given gauge boson is (in units where the AdS curvature scale is set to unity)

$$\frac{\Delta_J}{s} \ll 1, \tag{3.7}$$

where s can be defined in terms of boundary momenta.

Massive gauge bosons in AdS are dual to boundary currents that are not conserved. Conserved currents have vanishing divergence, which implies that certain descendant states are eliminated from the Hilbert space of the theory. In other words, they belong to the "short" representation of the conformal algebra, as there are fewer states than those associated with the "long" representation of a non-conserved current [60, 61]. One may combine short representations to obtain a long representation [62], which is analogous to supplying additional degrees of freedom to massless bulk gauge fields to make them massive. For a non-conserved current J, the limit in Eq. (3.7) reproduces the Ward identity at the boundary,

$$\partial \cdot \langle J \ldots \rangle \propto \langle \mathcal{O}_\pi \ldots \rangle \tag{3.8}$$

for a scalar primary $\mathcal{O}_\pi$. We find that the naively descendant operator, $\partial \cdot J$, is approximately primary when computing conformal correlators. Moreover, the non-conserved current is approximately a functional of a primary scalar,

$$J_\mu \propto \partial_\mu \partial^{-2} \mathcal{O}_\pi. \tag{3.9}$$

We generalize this result to show the dominant contribution to massive spin-l interactions comes from lower spin Goldstone-like fields. At sufficiently high energies, the tower of equivalences collapses to leave a dominant scalar contribution. On the CFT side, this takes the form

$$\mathcal{O}_{\mu_1 \ldots \mu_l} \approx \partial_{\mu_1} \ldots \partial_{\mu_l} \left(\partial^{-2}\right)^l \mathcal{O}. \tag{3.10}$$

The above results come from the *bulk* ET. It is a natural continuation, then, to see if the ET can be extracted purely in terms of CFT operators and correlation functions, without any reference to the specifics of the bulk interacting theory. Since conformal field theories do not admit an S-matrix in the traditional sense, a natural concern might be how the high energy limit is extracted in terms of dual operators. Moreover, the ET is intimately tied with the polarization vectors associated with massive gauge bosons. Are there analogs of flat-space polarization vectors that can be contracted with conformal correlators of tensor currents? In order to make any progress, we will turn to a completely model-independent, bottom-up approach to CFTs in the form of the operator product expansion ('OPE') [63], where one may expand in the distance between two operators

$$\mathcal{O}(x)\mathcal{O}(y) \sim \sum_{\mathcal{O}} \lambda_{\mathcal{O}} C(x-y, \partial_y)\mathcal{O}(y). \tag{3.11}$$

The sum will generally run over all primary operators present in the theory (i.e. the scaling dimensions Δ and spins ℓ of these states) and the coefficients λ of the above algebra specify the dynamics of the theory. The function $C(x-y, \partial_y)$ is completely fixed by conformal invariance, modulo an overall constant. In a CFT, such an algebra has a finite radius of convergence and becomes particularly powerful in the analysis of correlation functions through the bootstrap program [64–69]. The OPE also implies that one is always able to reduce higher point correlation functions to those fixed by conformal symmetry [70, 71]. For example, applying the OPE twice to a four point function of scalars yields

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \\ &\quad\times\frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\sum_{\mathcal{O}}\lambda_{\mathcal{O}}^{12}\lambda_{\mathcal{O}}^{34}G_{\mathcal{O}}(u,v) \\ &\equiv\sum_{\mathcal{O}}\lambda_{\mathcal{O}}^{12}\lambda_{\mathcal{O}}^{34}W_{\mathcal{O}}(u,v),\end{aligned}\tag{3.12}$$

where Δ_i are the scaling dimensions of the external operators, and $G_{\mathcal{O}}$ are the global conformal blocks and denote the contribution of a given exchanged primary and its descendants to the four-point function. In other words, they are the projection of $\mathcal{O}_{\Delta,\ell}$ onto the four point function. The general form of these conformal blocks was only recently determined by Dolan & Osborn [72, 73] in two and four dimensions for external scalar operators. In order to account for the tensor structures inherent in our analysis (since our external operators will be currents), we will turn to the formalism recently developed in [1] and [2]. In this approach, the role of the polarization vectors is most aptly played by auxiliary vectors Z^{A_i} in embedding space that generate conformally invariant scalar correlation functions out of those that manifestly involve operators with spin. The usage of this formalism to compute the conformal blocks is contingent on the assumption that the exchanged operators in the OPE are symmetric and traceless.

Under this caveat, our main result on the CFT side will come from looking at the divergence of a four point function consisting of a single spin-1 non-conserved current and three other scalar operators. We will show that when this correlation function is decomposed in terms of its conformal blocks, the blocks themselves satisfy an ET when the twists $\tau = \Delta - l$ of the exchanged operator are large compared to the dimension of the current (and $\Delta \gg l$). This result is illustrated in Fig. 3.2. We find that this relation holds up to $\mathcal{O}(1)$ functions that depend only on the coordinates and dimensions of the external operators. As stated, in order to do a partial wave analysis of current correlation functions, we will extensively use two tools: the embedding or "null cone" formalism and the index-free formalism developed in [1] and [2], which we will briefly review in §3.6.2.1.

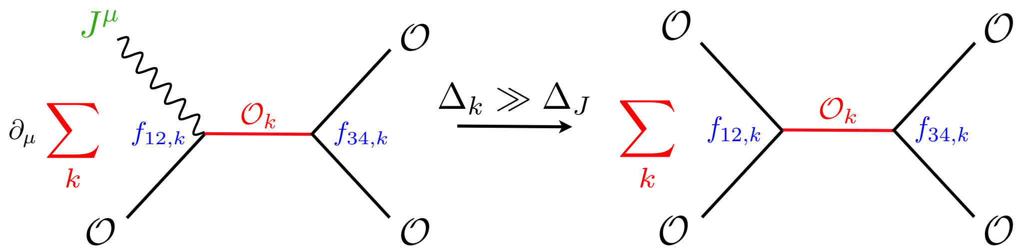

Figure 3.2: The CFT ET is schematically illustrated above. The four point function of a spin-1 current and three scalars can be decomposed as a sum over exchanged operators (the conformal blocks). We find that when we the twists of these operators are large compared to the dimension of the current, the blocks associated with the current four point function satisfy an ET (they become scalar blocks, as shown on the right hand side).

This chapter is organized as follows: in §3.2, we review how the Schwinger-Dyson relations work for a general spontaneously broken theory. In §3.3, we review how these relations, in conjunction with the LSZ formula, give rise to the familiar ET. In §3.4, we then generalize this proof to AdS by expanding gauge fields in terms of mode functions. In §3.5, we generalize the AdS ET for arbitrary spin fields. In §3.6, we demonstrate how the ET works in two and four dimensional CFTs; the former is the maximally simple case and serves as a warm-up to the latter, which requires the aforementioned index-free formalism. Finally, we conclude with §3.7.

3.2 Schwinger-Dyson Relations

In what follows, it will be necessary to establish the most general form of the Lagrangian for a broken gauge theory in a general spacetime. We will derive a relationship between correlation functions of gauge fields and associated Goldstone fields arising from this Lagrangian. To proceed, we recognize the Goldstone bosons must be derivatively coupled in the Lagrangian; additionally, in the absence of a gauge fixing term, gauge invariance must still be preserved at the Lagrangian level. This forces the Goldstone sector of the Lagrangian to have the form

$$\mathcal{L} \supset \mathcal{L}_{GS}\left[(A^a - m_A^{-1} D\pi^a)^2\right]; \tag{3.13}$$

$$\mathcal{L}_{GS} = \frac{1}{2} m_A^2 (A^a - m_A^{-1} D\pi^a)^2 + \mathcal{L}_{int}\left[\frac{1}{2}(A^a - m_A^{-1} D\pi^a)^2, \chi\right] \tag{3.14}$$

where the gauge algebra index, a, runs over only the values associated with the broken gauge bosons, m_A is the gauge boson's mass, and D is the gauge covariant derivative. Here, χ is used to represent any additional fields that may interact with the gauge and Goldstone bosons. The interaction Lagrangian, $\mathcal{L}_{int}$, must vanish when fields other than the gauge and Goldstone bosons vanish: $\mathcal{L}_{int}\left[\frac{1}{2}(A^a - m_A^{-1}D\pi^a)^2, 0\right] = 0$. To $\mathcal{L}$, we add the R_ξ gauge fixing and ghost terms for full generality

$$\mathcal{L}_{GF} + \mathcal{L}_{GH} = -\frac{1}{2}\xi^{-1}G^2 - \bar{c}\frac{\delta G}{\delta\theta}c; \tag{3.15}$$

$$G \equiv g^{MN}\nabla_M A_N^a - \xi m_A \pi^a \implies \frac{\delta G}{\delta\theta} = \nabla^M D_M + \xi m_A, \tag{3.16}$$

and a generic interaction Lagrangian, $\mathcal{L}_{G,int}(A^a)$, for any additional interactions the gauge fields may have. This yields the sub-Lagrangian for the broken gauge sector,

$$\begin{aligned}\mathcal{L} \supset &\left[-\frac{1}{4}(F^a_{broken})^2 + \frac{1}{2}m_A^2(A^a)^2 - \frac{1}{2}\xi^{-1}\left(g^{MN}\nabla_M A_N^a\right)^2\right] + \left[\frac{1}{2}(D\pi^a)^2 - \frac{1}{2}\xi m_A^2(\pi^a)^2\right] \\ &+ \mathcal{L}_{int}\left[\frac{1}{2}(A^a - m_A^{-1}D\pi^a)^2, \chi\right] + \mathcal{L}_{G,int}[A^a] - \bar{c}\frac{\delta G}{\delta\theta}c. \end{aligned}\tag{3.17}$$

As a formality, the Schwinger-Dyson equations for the gauge and Goldstone bosons are given by

$$\begin{aligned}[\nabla_N\nabla^M - \xi^{-1}\nabla^M\nabla_N - (\nabla^{2M}_N + m_A^2\delta_N^M)]\langle TA^{aN}\dots\rangle &= \langle T(A^{aM} - m_5^{-1}\partial^M\pi^a)\mathcal{L}'_{int}\dots\rangle \\ &+ \langle J^M\rangle + C_G, \end{aligned}\tag{3.18}$$

$$\begin{aligned}(\nabla^2 + \xi m_A^2)\langle T\pi^a\dots\rangle &= -m_A^{-1}\nabla_M\langle T(A^{aM} - m_5^{-1}\partial^M\pi^a)\mathcal{L}'_{int}\dots\rangle \\ &+ C_{GS}, \end{aligned}\tag{3.19}$$

where the C's are contact terms, $\mathcal{L}'_{int}$ is the derivative of $\mathcal{L}_{int}$ with respect to its first argument, and J^M is a conserved current to which the gauge fields couple, $J^{aM} = \frac{\partial}{\partial A^a{}_M}\left[\mathcal{L}_{G,int}[A^a] + \mathcal{L}_{GH}\right]$. The '$\ldots$' include other field operators.

The relevant consequence of these equations for the ET is found by taking the covariant divergence of Eq. (3.18), yielding, up to contact terms,

$$m_A^{-1}\nabla_M[\nabla_N\nabla^M - \xi^{-1}\nabla^M\nabla_N - (\nabla^{2M}_{\;\;N} + m_A^2\delta^M_N)]\langle TA^{aN}\ldots\rangle = -[\nabla^2 + \xi m_A^2]\langle T\pi^a\ldots\rangle, \tag{3.20}$$

where T is the time-ordering operator. It is worthwhile noting that m_A here is actually the physical mass, not simply the bare mass that naively appears in the Lagrangian. The significance of the above equation is that there exists projection operators that relate correlation functions involving gauge fields to those involving Goldstone modes. Generally, wave function renormalization factors must also appear, but it has been shown that a renormalization scheme can always be chosen so that they will cancel in Eq. (3.20) [74].

In this section, we made the gauge index explicit. However, it will not affect any future results, so we will drop it simply as a notational convenience.

3.3 ET in Flat Space

3.3.1 Proof for External Legs with Large Momenta

Consider an arbitrary S-matrix element involving a longitudinally polarized gauge boson with spatial momentum $\vec{p}$ and mass m_A in flat space written in terms of correlation functions of fields per the LSZ reduction scheme:

$$\langle \Psi_F | A_L, \vec{p}; \Psi_I \rangle = i \int d^{d+1}x \, \epsilon_{L,\mu} e^{-ip\cdot x} \left[\partial^2 \delta^\mu_\nu + m_A^2 \delta^\mu_\nu - (1-\xi^{-1}) \partial^\mu \partial_\nu \right] \langle T A^\nu \ldots \rangle, \tag{3.21}$$

where $p_0 = \sqrt{\vec{p}^2 + m_A^2}$. The states $\Psi_{I,F}$ are assumed to be created from functions of creation and annihilation operators of various fields, which are included in the '$\ldots$' on the RHS of Eq. (3.21). The differential operator acting on the correlation function can be identified as the one on the LHS of Eq. (3.18). The contact terms on the RHS contribute to the identity part of the S-matrix element. This identity piece trivially satisfies the ET since it must be the same for scalars and vectors of any masses up to the necessary minus sign for each freely propagating external leg, which arises from the polarization normalization condition $\epsilon_s(p) \cdot \epsilon_s(-p) = -1$. We thus concern ourselves with only the role of interacting processes in the ET and ignore contact terms henceforth.

In the limit $\vec{p}^2 \gg m_A^2$, we find $\epsilon_{L,\mu} = m_A^{-1} p_\mu + \mathcal{O}(m_A^2/\vec{p}^2)$. Inserting this into Eq.

(3.21) we obtain

$$\langle \Psi_F | A_L, \vec{p}; \Psi_I \rangle = \int d^{d+1}x \, e^{-ip\cdot x} m_A^{-1} \partial_\mu \left[\partial^2 \delta^\mu_\nu + m_A^2 \delta^\mu_\nu - (1-\xi^{-1}) \partial^\mu \partial_\nu \right] \langle T A^\nu \dots \rangle \tag{3.22}$$

$$= \int d^{d+1}x \, e^{-ip\cdot x} \left[\partial^2 + \xi m_A^2 \right] \langle T \pi \dots \rangle = -i \langle \Psi_F | \pi, \vec{p}; \Psi_I \rangle, \tag{3.23}$$

where $\langle \Psi_F | A_L, \vec{p}; \Psi_I \rangle$ is the process involving external gauge bosons and $\langle \Psi_F | \pi^a, \vec{p}; \Psi_I \rangle$ is the S-matrix element for the same process in which the longitudinally polarized gauge boson has been replaced with its corresponding Goldstone boson. The second to last line was obtained from the preceding one by using Eq. (3.20). The equivalence between the last two lines requires that $p^2 \to 0$. While the masses of the Goldstone bosons and gauge bosons generally differ, their energies are dominated by momentum and are thus both approximately massless.

Note that the statement $\vec{p}^2 \gg m_A^2$ is not Lorentz-invariant while the equivalency of the S-matrices themselves must be. The appropriate Lorentzian generalization of this frame-dependent limit should be $\frac{m_A^2}{s} \ll 1$, where s is the center of mass energy. Since s involves the energies and momenta of all the gauge bosons in the scattering process, this suggests we may make the replacement $A \to \pi$ for all longitudinally polarized gauge bosons as long as this limit is satisfied. To confirm that this is indeed the correct Lorentzian generalization, consider a general scattering process in the center of mass frame with a number of incoming particles, with energies $p_{0,i} = \sqrt{\vec{p}_i^2 + m_i^2}$. In this frame, $\sum_i \vec{p}_i = 0$, so $s = \sum_{j,i} p_{0,i} p_{0,j}$. Considering the case in which $s \sim m_A^2$, increasing s by a factor c sufficiently large that $\frac{m_a^2}{s} \ll 1$, each p_0 must increase by a factor $\sqrt{c}$. This can only be accomplished by increasing $|\vec{p}_i|$ by a correspondingly large factor such that $\vec{p}_i^2 \gg m_i^2$. The LSZ formula for this process with n gauge bosons will contain $\epsilon_L(\vec{p}_1) \otimes \epsilon_L(\vec{p}_2) \otimes \cdots \otimes \epsilon_L(\vec{p}_n) \approx m^{-n} p_1 \otimes \cdots \otimes p_n$. As was shown, the momenta become derivatives of the scalar wave function in the integral in this frame. This set of derivatives is Lorentz covariant. Transforming to a different frame simply boosts each derivative to a derivative in the new frame, thereby confirming this limit is the correct Lorentzian generalization.

3.3.2 The Connection of the Exchange Operator to the ET

In the previous section, the ET was demonstrated in the scenario that the center of mass energy is large. Broken gauge theories clearly admit couplings such that scattering amplitudes have appreciable support at the low energies, in which case the ET is only true for S-matrix elements in the high energy limit. If, instead, the theory contains couplings such that interactions occur only at large energies anyway, then the ET should be automatically satisfied at any momentum scale of the external legs since the identity part of the S-matrix trivially satisfies the ET and interactions would be irrelevant until momentum scales that satisfy the ET are reached anyway. If interactions are mediated by only by particles with mass m_{other} such that $\frac{m_A^2}{m_{\text{other}}^2} \to 0$, then the theory satisfies this coupling criteria. This follows since poles in the vertex function for scattering processes would be pushed high enough to render interactions negligible except at large energies, $s \sim m_{\text{other}}^2$.

As an example, consider a gauge theory coupled to a Higgs sector with a very large mass and Yukawa couplings to an uncharged scalar:

$$\mathcal{L} \supset -\frac{1}{4}F^2 + |D\Phi(h)|^2 - \frac{1}{2}m_H^2 h^2 - \frac{1}{2}yh\phi^2 \tag{3.24}$$

with $\frac{m_A^2}{m_H^2} \ll 1$ and where $\Phi(h)$ is in the fundamental representation of the gauge group.

This Lagrangian admits the potential four-point interacting process $A_L A_L \to \phi\phi$, which is diagrammatically represented in Fig. 3.3.

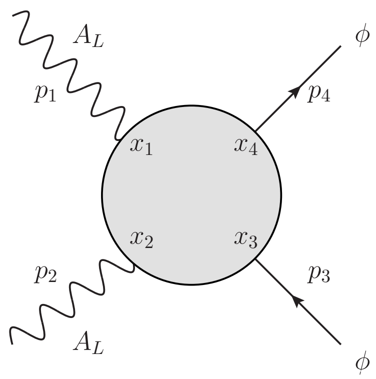

Figure 3.3: The depiction of a general process sending two incoming longitudinally polarized gauge bosons to two outgoing scalars. The vertex function variations $x_1, \ldots, x_4$ are integrated over in the LSZ formula.

The LSZ formula for this process reads

$$S[A_L] \equiv \langle \vec{p}_3; \vec{p}_4 | \vec{p}_1, L; \vec{p}_2, L \rangle = -\int d^{d+1}x_1 \ldots d^{d+1}x_4 \, \epsilon_{L,\mu_1}(p_1)\epsilon_{L,\mu_2}(p_2) e^{-i\sum_{i=1}^{4} p_i \cdot x_i} \times \Gamma^{\mu_1\mu_2}(x_1, x_2, x_3, x_4) \tag{3.25}$$

where

$$\Gamma^{\mu_1\mu_2}(x_1, x_2, x_3, x_4) \equiv {}_A D^{2\mu_1}{}_{\nu_1} \, {}_A D^{2\mu_2}{}_{\nu_2} \, {}_\phi D^2{}_1 \, {}_\phi D^2{}_2 \langle T A^{\nu_1}(x_1) A^{\nu_2}(x_2) \phi(x_3) \phi(x_4) \rangle \tag{3.26}$$

is the vertex function (amputated Green function) with

$${}_A D^{2\mu}_{\ \nu} \equiv \left[(\partial^2 + m_A^2)\delta^\mu_\nu - (1 - \xi^{-1})\partial^\mu \partial_\nu \right], \tag{3.27}$$

$${}_\phi D^2 \equiv \left[\partial^2 + m_\phi^2 \right]. \tag{3.28}$$

At the level of interactions, the emergence of the ET for large poles in $\Gamma^{ab,\mu_1\mu_2}$ is most easily demonstrated at leading order, as shown diagrammatically in Fig. 3.4.

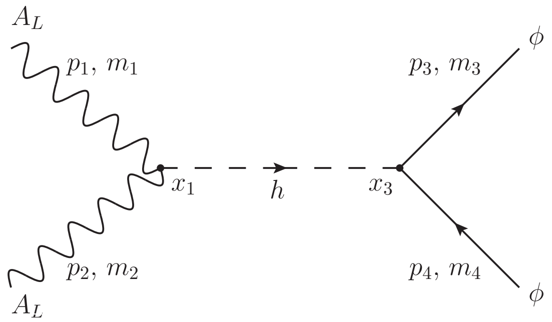

Figure 3.4: The leading order contribution to the interacting piece of the S-matrix for the process depicted in Fig. 3.3.

with the interaction vertex function given by

$$\Gamma^{\mu_1\mu_2} \propto \int d^{d+1}k \frac{i}{k^2 - m_H^2 + i\epsilon} e^{-ik\cdot(x_3 - x_1)}, \tag{3.29}$$

Inserting Eq. (3.29) into Eq. (3.25) and performing the integrals over x_1 and x_3 yields a momentum-conserving delta function $(2\pi)^{d+1}\delta^{d+1}(p_1+p_2-k)$ that forces the constraint $k^2 = s$. For sufficiently large m_H^2, the effective operator $A^2\phi^2$ is extremely suppressed, with scattering amplitude $\mathcal{M} \propto \frac{ym_A}{m_H^2} \sim 0$, implying that the exchange essentially does not occur, leaving only the identity piece and trivially satisfying the ET. The exchange becomes relevant when $s \sim m_H^2$, which corresponds to the condition $\frac{m_A^2}{s} \ll 1$, at which point the ET is satisfied anyway.

Evaluating correlation functions of fields involves integrating objects $S[A, \dots]$ over the external momenta, which will generally include some scale for which the ET does not hold. The above results, however, open the possibility of replacing the longitudinal degree of freedom with the derivative of a scalar for theories that require a large invariant mass to excite exchanges.

3.4 ET in AdS

Expressing S-matrix elements using the LSZ formula in previous sections admitted a simple derivation in which the equivalence of the derivative of the scalar wave function and the vector wave function in a particular limit implies the ET from the Schwinger-Dyson equations. In AdS, the initial and final states which are related by the S-matrix are prepared by acting on the vacuum state with CFT operators [75–77]. Moreover, the LSZ formula becomes a generalization of the AdS/CFT dictionary that relates correlators of boundary creation and annihilation operators to integrals of bulk fields. For parallelism and succinctness, these correlators will be referred to as matrix elements since they should reproduce the S-matrix in the flat space limit. These matrix elements are related to momentum space conformal correlators, which will be shown within this section and whose utility has been shown in [78] and [79]. By utilizing this generalization, a nearly identical derivation of the ET for matrix elements that follows from the equivalence of wave functions can be made in AdS.

3.4.1 AdS Wave Functions

The LSZ formula for a given external leg in some process in flat space reads schematically as

$$S[\phi, i] = \int dx N_i f_i(x) D^2 \langle T\phi \dots \rangle, \tag{3.30}$$

where S is the S-matrix element involving an external ϕ leg labeled by quantum numbers i, $f_i(x)$ is the (normalized) wave function for the particle ϕ in this process, N_i is the state normalization for that leg, and D^2 is a differential operator inverse of the ϕ propagator. The scalar and vector wave functions labeled by momentum are simply $f(\vec{p}, x) = \frac{1}{\sqrt{2p_0}} e^{-ip\cdot x}$ and $h_{s\mu}(\vec{p}, x) = \frac{1}{\sqrt{2p_0}} \epsilon_{s\mu}(\vec{p}) e^{-ip\cdot x}$, respectively, and the momentum state normalization is given by $|\vec{p}, s\rangle = \sqrt{2p_0} a_s^\dagger(\vec{p})|\Omega\rangle \implies N_{\vec{p}} = \sqrt{2p_0}$. To obtain an AdS ET using the LSZ formula as a generalization of the AdS/CFT dictionary, we need to specify appropriate AdS scalar and vector wave functions and state normalizations. In Poincaré patch coordinates, it is natural to expand the gauge fields as

$$A_M(x,z) = \sum_{s=1}^{d+1} \int d^d\vec{p} \int_0^\infty dm \left[a_s(\vec{p},m) h^\dagger_{s,M}(\vec{p},m;x,z)(x) + h.c. \right] \tag{3.31}$$

and (real) scalar fields as

$$\phi(x,z) = \int d^d\vec{p} \int_0^\infty dm \left[b(\vec{p},m) f^\dagger(\vec{p},m;x,z) + h.c. \right] \tag{3.32}$$

where

$$f(\vec{p},m) = \frac{\sqrt{m}}{\sqrt{2p_{m0}}} z^{\frac{d}{2}} J_{\Delta_\phi - \frac{d}{2}}(mz) e^{-ip_m\cdot x}, \tag{3.33}$$

$$h_{s,M}(\vec{p},m) \underset{s\neq z,\xi}{=} \begin{cases} 0, & M=z \\ \frac{\sqrt{m}}{\sqrt{2p_{m0}}} \epsilon_{s,\mu}(\vec{p}) z^{\frac{d}{2}-1} J_{\Delta_J - \frac{d}{2}}(mz) e^{-ip_m\cdot x}, & M=\mu \end{cases}, \tag{3.34}$$

$$h_{z,M}(\vec{p},m) = \begin{cases} \frac{\sqrt{m}}{\sqrt{2p_{m0}}} z^{\frac{d}{2}} J_{\Delta_J - \frac{d}{2}}(mz) e^{-ip_m\cdot x}, & M=z \\ -i\frac{\sqrt{m}}{\sqrt{2p_{m0}}} \frac{p_{m\mu}}{m^2} \left[m z^{\frac{d}{2}} J_{\Delta_J - \frac{d}{2}+1}(mz) - [\Delta_J - (d-1)] z^{\frac{d}{2}-1} J_{\Delta_J - \frac{d}{2}}(mz) \right] e^{-ip_m\cdot x}, & M=\mu \end{cases} \tag{3.35}$$

in which the particles' masses have been replaced by the their scaling dimensions: $m_A^2 \to [\Delta_J - (d-1)](\Delta_J - 1)$ and $m_\phi^2 \to \Delta_\phi(\Delta_\phi - d)$. The set of vectors $\{\epsilon_{s,\mu}\}$ are the usual Lorentzian polarizations. The parameter p_m is just the Lorentzian momentum with mass $|p_m| = m$. Here, we are letting m be a degree of freedom over which we integrate instead of the traditional p_0 typically found in the literature. This serves two purposes. First, it naturally imposes the restriction that the boundary momenta be time-like instead of requiring the Fourier transform be only over positive squared norms. Second, it lends itself to considering a foliation of AdS over space-like Cauchy surfaces at constant Poincaré patch time, which is more in keeping with the flat space approach. After our analysis, it should be rather clear that our foliation did not matter and we will ultimately state the results in terms of boundary d-momenta, p, instead of $\vec{p}$ and m.

For completeness, it should be noted that the above listed vector wave functions do not account for the $d+1$ total polarizations that must be summed over in Eq. (3.31) to account for all (nominal) degrees of freedom. There is an additional, unphysical wave function associated with the divergence of A that takes the form $h_{\xi,M} = \partial_M \nabla^{-2} f_{\Delta_\phi \to \Delta_\xi}$ that will be inconsequential in the following sections. It will also be useful to explicitly write the derivative of the scalar wave function:

$$\partial_M f(\vec{p}, m) = \begin{cases} -\frac{\sqrt{m}}{\sqrt{2p_{m0}}} \frac{p_{m\mu}}{m^2} \left[m z^{\frac{d}{2}} J_{\Delta_J - \frac{d}{2}+1}(mz) - \Delta_J z^{\frac{d}{2}-1} J_{\Delta_J - \frac{d}{2}}(mz) \right] e^{-ip_m \cdot x}, & M = z \\ -i\frac{\sqrt{m}}{\sqrt{2p_{m0}}} p_{m\mu} z^{\frac{d}{2}} J_{\Delta_\phi - \frac{d}{d}}(mz) e^{-ip_m \cdot x}, & M = \mu \end{cases} . \tag{3.36}$$

In order to preserve the conformal invariance of the inner products of states, we choose the normalization $|\vec{p}, m, s\rangle = \sqrt{2p_0 m}\, a_s^\dagger(\vec{p}, m)|\Omega\rangle$. Under this choice in basis wave functions and state normalization, the LSZ-like formula for a matrix element reads

$$S[A,\vec{p},m,s] = i\int d^4x \int_0^\infty dz\, N(\vec{p},m) h_{s,M}(\vec{p},m)\, {}_A D^M_N \langle T A^\nu \dots\rangle, \tag{3.37}$$

$$S[\phi,\vec{p},m] = i\int d^4x \int_0^\infty dz\, N(\vec{p},m) f(\vec{p},m)\, {}_\phi D^2_{x_i} \langle T \phi \dots\rangle, \tag{3.38}$$

where

$${}_A D^{2M}_N = \left[\left(\nabla^{2M}_{\ \ N} + (\Delta_J - (d-1))(\Delta_J - 1)\delta^M_N\right) - \nabla_N \nabla^M + \xi^{-1}\nabla^M \nabla_N\right], \tag{3.39}$$

$${}_\phi D^2 = [\partial^2 + \Delta_\phi(\Delta_\phi - d)], \tag{3.40}$$

and $N = \sqrt{2p_{m0}m}$.

3.4.2 The AdS ET

Continuing in parallel with the methods from §3.3, we must demonstrate the equivalency of the LSZ integrals under the exchange $h_s \to \partial f$ for some particular spin index, 's', in some particular limit to show that a given external vector can be replaced with a scalar. In other words, we would like to determine longitudinal polarizations in terms of the mode functions. To understand which spin is relevant, we can briefly consider the dual boundary current and determine which spin corresponds to the degree of freedom introduced by breaking the gauge symmetry in the bulk. To understand what the analogous limit to large momentum in §3.3.1 is, we can work in the reverse order of the flat space sections and first examine the four-point matrix element in AdS for the same process that was considered in §3.3.2 in the limit that the scaling dimension of the exchange operator (i.e. the exchanged mass) is large. This should then reveal the constraints on the quantum numbers of incoming/outgoing states for the ET to hold.

Since breaking the gauge symmetry generates a mass for the gauge boson, its dual current should not be conserved. This divergence degree of freedom of the current must thus play the role of the longitudinal degree of freedom in the flat space case, so we should expect that the degree of freedom of the gauge boson that corresponds to the divergence of the current when carried to the boundary is precisely the analogue to the longitudinal degree of freedom.

The gauge boson is identified with its dual current by

$$\begin{aligned} J_\mu^a &= \lim_{z\to 0} \frac{F^a_{z\mu}}{z^{\Delta_J-2}} \\ &= \lim_{z\to 0} \frac{\partial_z A^a_\mu - \partial_\mu A^a_z}{z^{\Delta_J-2}} \\ &= \frac{\Gamma(\Delta_J)}{\Gamma(\Delta_J-1)} \lim_{z\to 0} \frac{\bar{A}^a_\mu}{z^{\Delta_J-1}}, \end{aligned} \tag{3.41}$$

where

$$\bar{A}^a_M \equiv \left[\delta^N_M - \partial_M \nabla^{-2} \nabla^N\right] A^a_N \tag{3.42}$$

is the gauge field with the gauge dependence projected out. The replacement $A \to \bar{A}$ can be made since it leaves the field strength tensor unchanged and is useful as it allows us to ignore the $s = \xi$ wave function index in the expansion of A. For notational tractability, we will redefine $J \to \frac{\Gamma(\Delta_J)}{\Gamma(\Delta_J-1)} J$ to eliminate the gamma factors. The divergence of the current is then

$$\begin{aligned} \partial \cdot J^a &= \lim_{z\to 0} \frac{\partial \cdot \bar{A}^a}{z^{\Delta_J - 1}} \\ &= \lim_{z\to 0} \frac{1}{z^{\Delta_J - 3}} \left[\nabla_M \bar{A}^{aM} - \nabla_z \bar{A}^{az}\right] \\ &= \lim_{z\to 0} \frac{1}{z^{\Delta_J - 1}} \left[z^{d-1}\partial_z \left(\frac{1}{z^{d-1}} \bar{A}^a_z\right)\right]. \end{aligned} \tag{3.43}$$

Since the only wave function that contributes to $\bar{A}_z$ is h_z, $s = z$ must play the role of bulk longitudinal polarization. This result may appear naively gauge dependent since we might expect symmetry breaking to generate a z-component only in the $A_z = 0$ gauge and not a general ξ gauge. It might also seem unusual that A_M possessing a nontrivial z-component seems unrelated to longitudinal propagation in the flat space limit. We know, however, that the longitudinal polarization in flat space is the only polarization with a component in the time direction, so it is not entirely surprising that the relevant wave function is the only one with non-vanishing components in a preferred direction.

Having determined the appropriate s index, we now turn to the constraints imposed on the quantum numbers of external states in order to excite a four-point process involving heavy exchanges as depicted in Fig. 3.5.

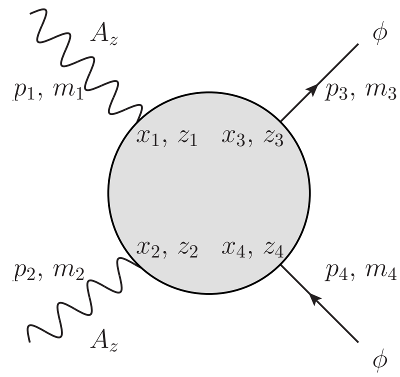

Figure 3.5: The general four-point diagram for gauge bosons "incoming" gauge bosons at different times (as defined by creation operators) ending with "outgoing" scalars at different times (as defined by annihilation operators). The arguments of the vertex function $\{x_i, z_i\}$ are integrated over in the LSZ formula.

The complete LSZ formula for this process is

$$S[A_z] \equiv \langle \vec{p}_3, m_3; \vec{p}_4, m_4 | \vec{p}_1, m_1; \vec{p}_2, m_2 \rangle = \left(\prod_{i=1}^{4} \int d^4x_i \int_0^\infty dz_i \sqrt{g(z_i)} \right)$$
$$\times \left(\prod_{i=1}^{2} N_A(\vec{p}_i, m_i) h_{z,M_i}(\vec{p}_i, m_i; x_i, z_i) \right) \left(\prod_{i=3}^{4} N_\phi(\vec{p}_i, m_i) f_\phi(\vec{p}_i, n_i; x_i, z_i) \right) \Gamma^{M_1 M_2}_{AA\phi\phi}, \tag{3.44}$$

where the vertex function, $\Gamma^{M_1 M_2}_{AA\phi\phi}$, is defined in the analogous way to what was encountered in Eq.(3.26). The important consequence of the Schwinger-Dyson relations now amount to

$$\nabla_{M_1} \nabla_{M_2} \Gamma^{M_1 M_2}_{AA\phi\phi} = \left[(\Delta_J - (d-1))(\Delta_J - 1) \right] \Gamma_{\pi\pi\phi\phi}, \tag{3.45}$$

where $\Gamma_{\pi\pi\phi\phi}$ is the scalar vertex function. Expectedly, when $\Delta_J = (d-1)$, the gauge boson is massless and the usual Ward identities are satisfied.

As before, the identity part of the matrix trivially satisfies the ET and we turn to the same hypothetical leading order contribution for the exchange to make the notion of large scattering energies concrete. The vertex function then takes the form

$$\Gamma^{M_1 M_2}_{AA\phi\phi} \propto \int_0^\infty dn\, n(z_1 z_3)^{\frac{d}{2}} J_{\Delta-\frac{d}{2}}(nz_1) J_{\Delta-\frac{d}{2}}(nz_3) G_n(x_3 - x_1), \tag{3.46}$$

where G_n is the usual lorentzian propagator for a scalar of mass n and Δ is the scaling dimension of the exchanged scalar. A heavy/energetic exchange corresponds to large Δ: $\frac{\Delta_J}{\Delta} \ll 1$. This makes sense since the scaling dimensions of exchanged operators in conformal theories can be thought of as a measurement of the center of mass energy. We will elaborate on this point when we discuss the ET on the side of the CFT.

For $nz \ll \sqrt{\Delta}$, $J_{\Delta-\frac{d}{2}} \approx \frac{1}{\Gamma(\Delta-\frac{d}{2}+1)} \left(\frac{nz}{2}\right)^{\Delta-\frac{d}{2}}$, which is very strongly suppressed by Δ. So $J_{\Delta-\frac{d}{2}}(nz)$, and consequently the entire LSZ integral[2], is dominated by large nz behavior for large Δ. Then only either n or z needs to be large for the exchange to be relevant. Since both parameters are integrated over in the LSZ integral, we consider the two relevant regions of parameter space in which one remains finite and the other is large.

For the first region in which z is finite and $n \to \infty$, the integral in Eq. (3.46) is dominated by large n. The situation then closely resembles the flat space case: sufficiently large Δ pushes n^2, and consequently the poles in G_n, enough to render the exchange negligible except for $s \approx n^2$. In turn, this requires large $\vec{p}$'s and m's for this part of integration space to contribute to the exchange, which demands that

[2]This follows since any Bessel function $J_\alpha(x)$ dies more quickly as $x \to 0$ than $x \to \infty$

the argument mz in the vector wave functions be large. If the $|p|$'s are finite, then the contribution of this region of integration space is negligible; if they are large, then $mz \approx \sqrt{\Delta} \gg \sqrt{\Delta_J}$, and the Bessel functions in the wave functions take their asymptotic forms for large arguments: $J_\alpha(mz) \approx \sqrt{\frac{2}{\pi mz}} \cos\left(mz - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right)$.

For the second case in which n is finite and $z \to \infty$, the story is much more trivial. Since $nz \approx \sqrt{\Delta}$ is large, and the poles in G_n set $n = \sqrt{s}$ after integrating over n, we conclude $mz \approx \sqrt{\Delta} \gg \sqrt{\Delta_J}$. The wave functions then assume the same asymptotic form as the previous case.

To confirm that the scalar and $s = z$ vector wave functions effectively share the same large argument behavior, we explicitly compare the large mz behaviors of $h_{z,M}$ and $\partial_M f$ to find

$$h_{z,M} \underset{mz \gg \sqrt{\Delta_J - \frac{d}{2}+1}}{=} \begin{cases} \frac{\sqrt{m}}{\sqrt{2p_{m0}}} z^{\frac{d}{2}} \sqrt{\frac{2}{\pi mz}} \cos\left[\frac{\pi}{4} + \frac{\pi}{2}(\Delta_J - \frac{d}{2}) - mz\right] e^{-ip_m \cdot x}, & M = z \\ -i\frac{\sqrt{m}}{\sqrt{2p_{m0}}} \left(\frac{p_{m\mu}}{m^2}\right) mz^{\frac{d}{2}} \sqrt{\frac{2}{\pi mz}} \cos\left[\frac{\pi}{4} + \frac{\pi}{2}(\Delta_J - (\frac{d}{2} - 1)) - mz\right] e^{-ip_m \cdot x}, & M = \mu \end{cases}, \tag{3.47}$$

$$\partial_M f = \begin{cases} -\frac{\sqrt{m}}{\sqrt{2p_{m0}}} mz^{\frac{d}{2}} \sqrt{\frac{2}{\pi mz}} \cos\left[\frac{\pi}{4} + \frac{\pi}{2}((\Delta_\phi + 1) - \frac{d}{2}) - mz\right] e^{-ip_m \cdot x}, & M = z \\ i\frac{\sqrt{m}}{\sqrt{2p_{m0}}} \left(\frac{p_{m\mu}}{m^2}\right) m^2 z^{\frac{d}{2}} \sqrt{\frac{2}{\pi mz}} \cos\left[\frac{\pi}{4} + \frac{\pi}{2}((\Delta_\phi + 1) - (\frac{d}{2} - 1)) - mz\right] e^{-ip_m \cdot x}, & M = \mu \end{cases}. \tag{3.48}$$

We find that $h_z = -\frac{1}{m}\partial f$ for $\Delta_\phi = \Delta_J - 1$, which is unsurprising since scalar and vector twists differ by 1. We now only need to address the presence of a phase shift above.

The integral over the Lorentzian coordinates and n can be performed in the large mz limit in Eq. (3.44) to arrive at the following expression for the interacting piece of the matrix element:

$$S[A_z] \propto \int_0^\infty dz_1\, z_1^{\frac{d}{2}-2} \Bigg\{ (m_1 m_2 + p_1 \cdot p_2) \cos\left[(m_1 - m_2) z_1\right] \\ + (m_1 m_2 - p_1 \cdot p_2) \sin\left[(m_1 + m_2) z_1 - \left(\Delta_J - \frac{d}{2}\right)\pi\right] \Bigg\} J_{\Delta - \frac{d}{2}}(\sqrt{s} z_1). \quad (3.49)$$

For the second term in curly brackets, we can change the integration variable from z_1 to y via $z_1 = y + \frac{1}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi$ and split the integral into a piece over the region $y \in [-\frac{1}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi, 0]$ and another over the region $y \in [0, \infty]$. The first region contributes negligibly since $J_{\Delta-\frac{d}{2}}(\frac{\sqrt{s}}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi) \sim \frac{1}{\Gamma(\Delta - \frac{d}{2}+1)}$, and we are left with only the second region. The second region is still negligible until $\sqrt{s}y + \frac{\sqrt{s}}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi \approx \sqrt{\Delta}$, at which point $y \gg \frac{1}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi$, permitting us to drop the $\frac{\sqrt{s}}{m_1+m_2}(\Delta_J - \frac{d}{2})\pi$ term. We are then left with Eq. (3.49) without the $(\Delta_J - \frac{d}{2})\pi$ phase shift in the second term. Since constant phase shifts are unimportant in the LSZ integral when Δ is very large, the difference between the phases in ∂f and h_z are irrelevant.

We can thus replace a single external gauge boson with a scalar when

$$\frac{\Delta_J}{m^2} \ll 1, \quad (3.50)$$

where the AdS curvature scale is set to unity. At this juncture, it is worthwhile to comment on our use of m as a quantum number in our mode function expansion earlier. Since $p^2 = m^2$, we can simply make the replacements $m \to |p|$ and allow $p_0 = \sqrt{\vec{p}^2 + m^2}$ to be the label for the external quantum states. With this labeling, the above condition becomes

$$\frac{\Delta_J}{|p|^2} \ll 1, \tag{3.51}$$

and all gauge bosons may be replaced with scalars when

$$\frac{\Delta_J}{s} \ll 1, \tag{3.52}$$

for a center of mass energy s. The AdS ET can then be stated as

$$S[A_z] = \left[-\sqrt{(\Delta_J - (d-1))(\Delta_J - 1)}\right]^N \left(\prod_{i=1}^{N} |p_i|^{-1}\right) S[\pi] \tag{3.53}$$

when

$$\frac{\Delta_J}{\Delta} \ll 1 \qquad \text{or} \qquad \frac{\Delta_J}{s} \ll 1. \tag{3.54}$$

Up to this point, the relevance of the matrix elements, S, in the language of AdS/CFT has been unclear. Their physical significance is evident in the flat space limit (which is incidentally the relevant limit herein) as S-matrix elements [80, 81]), but it would be useful to understand them in the context of conformal correlators. We note that we may express the correlator

$$\langle \mathcal{O}(x) \dots \rangle = \int d^d y dw \sqrt{g(w)} G_\partial(x-y; z)\Gamma(y, z; \dots), \tag{3.55}$$

where G_∂ is the bulk-boundary propagator and Γ is the usual bulk vertex function of interest here [37, 58]. We may convolve the conformal correlator with some boundary source, j, to yield

$$\int d^d x j(x;p)\langle \mathcal{O}(x)\dots\rangle = \int d^d x d^d y dw \sqrt{g(w)} j(x;p) G_\partial(x-y;z)\Gamma(y,z;\dots). \quad (3.56)$$

We may choose $j(x;p)$ such that $\int d^d y j(x;p) G_\partial(x-y;z) = f(x,z;p)$ is a wave function used to define S. For this to be the case, the left hand side of Eq. (3.56) must be proportional to the Fourier transform of the conformal correlator to which Γ is relevant. For boundary currents, this proportionality factor must involve a projection of correlator onto a vector in the tangent bundle at the conformal boundary. Consequently, Eq. (3.53) can be interpreted as a statement about the relation between the Fourier transform of conformal correlators dual to the gauge and Goldstone fields.

3.4.3 Implications for Correlation Functions of J_μ

At the end of §3.3.2, we discussed how the fact that theories with sufficiently large masses of exchanged particles satisfy the ET at all external energy scales and thus open the possibility for a manifestation of the ET in correlation functions of fields, as opposed to S-matrix elements, to appear since such objects involve the sum of S-matrix-like objects over all energy scales. This should have strong implications for correlation functions of conformal current operators under the AdS/CFT program, which will be examined in this section.

The relationship between the divergence of the conformal current and the bulk gauge fields is established in Eq. (3.43), so the AdS ET should manifest through this scalar degree of freedom. By defining

$$\mathcal{O}_J \equiv (\Delta_J - (d-1))^{-1}\partial \cdot J, \tag{3.57}$$

we may write the current as

$$J_\mu = (\Delta_J - (d-1))\partial_\mu \partial^{-2}\mathcal{O}_J + J^0_\mu, \tag{3.58}$$

where

$$J^{0a}_\mu \equiv \left[\delta^\nu_\mu - \partial_\mu \partial^{-2}\partial^\nu\right] J^a_\nu \tag{3.59}$$

is the conserved part of the current. Expanding Eq. (3.43) in terms of creation/annihilation operators yields

$$\partial \cdot J = (\Delta_J - (d-1)) \int d^{d-1}\vec{p} \int_0^\infty dm \, \left[a_z f^\dagger_\partial + h.c.\right] \tag{3.60}$$

where f_∂ is the scalar wave function with the same scaling dimension as the gauge boson taken to the boundary,

$$f_\partial(\Delta_J, \vec{p}, m; x) = \frac{1}{\Gamma(\Delta_J - \frac{d}{2} + 1)} \frac{1}{\sqrt{p_{m0}}} \left(\frac{m}{2}\right)^{\Delta_J - \frac{d-1}{2}} e^{-ip_m \cdot x}. \tag{3.61}$$

As expected, $\partial \cdot J$ vanishes when $\Delta_J = d-1$, corresponding to $m_A = 0$.

While $\mathcal{O}^a_J$ is manifestly a scalar, it is a descendant in general theories since J^a itself is primary. However, when $\frac{\Delta_J}{\Delta} \ll 1$, Eq. (3.53) shows that in correlation functions we may make the replacement

$$a_z \to -m^{-1}\sqrt{(\Delta_J - (d-1))(\Delta_J - 1)}b_\pi, \tag{3.62}$$

where b_π is the creation/annihilation operator for the corresponding Goldstone boson, which should have a primary dual. Defining the dual to the Goldstone as

$$\mathcal{O}_\pi \equiv -\frac{1}{2\left(\Delta_J - \frac{d}{2} + 1\right)}\sqrt{(\Delta_J - (d-1))(\Delta_J - 1)}\int d^{d-1}\vec{p}\int_0^\infty dm\,\left[b_\pi f_\partial^\dagger(\Delta_\pi) + h.c.\right] \tag{3.63}$$

$$= \int d^{d-1}\vec{p}\int_0^\infty dm\,\left[a_z f_\partial^\dagger(\Delta_\pi + 1) + h.c.\right], \tag{3.64}$$

where the last line follows from Eqs. (3.61) and (3.62), we see $\mathcal{O}_\pi$ is the same as $\mathcal{O}_J$ for $\Delta_\pi = \Delta_J - 1$. This is again unsurprising given the relative scaling of vectors and scalars to the boundary.

Recall that the non-interacting part of the matrix elements trivially satisfies the AdS ET and that the interacting part is insensitive to the scaling dimensions of the external particles when $\frac{\Delta_J}{\Delta} \ll 1$. The operators $\mathcal{O}_J$ and $\mathcal{O}_\pi$ may then be identified, and the expression of the AdS ET under the AdS/CFT prescription is thus

$$\mathcal{O}_J \to \mathcal{O}_\pi. \tag{3.65}$$

Equivalently, we may state that $\mathcal{O}_J$ is approximately primary when computing correlation functions. The usual techniques for computing correlators of primary operators for both $\mathcal{O}_\pi$ and J^0_μ may thus be used in theories in which $\frac{\Delta_J}{\Delta} \ll 1$.

To recapitulate what was shown from bulk AdS considerations:

- The $s = z$ wave function was shown to be analogous to the longitudinal polarization.

- The AdS equivalent to large $|\vec{p}|$ in flat space was shown to be large $|p|$ or s.
- For large interaction energies, $S[A_z] \propto S[\pi]$.
- The manifestation of the AdS ET at the boundary is, essentially, the Ward identity: $\partial \cdot \langle J \ldots \rangle \propto \langle \mathcal{O}_\pi \ldots \rangle$.

3.5 Higher Spin AdS ET

We have shown in the previous sections that, in light of the Dyson-Schwinger equations, the ET results simply from the asymptotic equivalency of the the derivative of a scalar wave function and a vector wave function in the high-energy limit. This implies an equivalence theorem relating spin-l processes to lower spin processes can then be obtained by demonstrating that the spin-l wave functions are asymptotically equivalent to symmetrized derivatives of lower-spin wave functions in the high-energy limit. We construct such a theorem in this section.

To begin, we briefly review higher spin fields. We wish to consider a massive real rank l field, $\phi_{M_1 \ldots M_l}(x, z)$, with scaling dimension Δ_J in an AdS_{d+1} vacuum that is symmetric, traceless, and transverse:

$$\phi_{M_1 M_2 \ldots M_l} = \phi_{M_2 M_1 \ldots M_l} = \phi_{(M_i \leftrightarrow M_j)} \forall i, j \tag{3.66}$$

$$\nabla^{M_1} \phi_{M_1 \ldots M_l} = 0 \tag{3.67}$$

$$\phi^{M_1}{}_{M_2 \ldots M_l} = 0. \tag{3.68}$$

Enforcing these conditions ensures that ϕ is an irreducible spin-l representation of the AdS isometries, and is the spin-l generalization of projecting out the unphysical divergence in gauge theories. That is, the degrees of freedom that are projected out by these constraints should correspond to "gauge" degrees of freedom. Consequently, we may unambiguously take this field to the CFT_d boundary in a "gauge independent" way and identify the CFT dual in the Poincaré patch as

$$\mathcal{O}_{\mu_1\ldots\mu_l}(x) \equiv \lim_{z\to 0} \frac{1}{z^{\Delta_J - l}} \phi_{\mu_1\ldots\mu_l}(x, z). \tag{3.69}$$

Per usual, we may expand the ϕ field in terms of mode functions. Demanding that the mode functions be in the same representation of the AdS isometries as our field means the mode functions must satisfy the classical *free* equations of motion for ϕ[3],

$$\left[\mathcal{D}^2{}^{N_1\ldots N_l}_{M_1\ldots M_l} + (\Delta_J - l)(\Delta_J - (d-l))\delta^{N_1}_{M_1}\ldots\delta^{N_l}_{M_l}\right](\phi_{free})_{N_1\ldots N_l} = 0 \tag{3.70}$$

where $\mathcal{D}^2$ is a second order differential operator containing the Laplace-Beltrami operator, ∇^2, as a linear contribution within it. This suggests we can expect to parameterize the mode functions in the Poincaré patch using the momentum in the boundary coordinate directions. Aditionally, a symmetric, traceless, transverse field of rank l in $d+1$ dimensions that is generally massive has $D \equiv \begin{pmatrix} d+l \\ l \end{pmatrix} - 2$ degrees of freedom,

[3]The (linearized) differential operator appearing in the free classical equations of motion is simply the Casimir of the AdS algebra, with the ϕ mass playing the role of the weight of the representation.

which corresponds to D polarizations. Our expansion then takes the form

$$\phi_{M_1 \ldots M_l}(x,z) = \sum_{s=1}^{D} \int \text{đ}^d p \ \left[a_{s,p}(z)\, \varphi^{(l)}_{M_1 \ldots M_l}(s,p;x,z) + h.c. \right], \tag{3.71}$$

with the notation $\text{đ} = \frac{1}{2\pi} d$.

Since we are parameterizing our mode functions using momentum, it is natural to foliate AdS in the z-direction and define a z-independent inner product over function space such that our mode functions are orthonormal[4],

$$\langle \varphi^{(l)}(s,p), \varphi^{(l)}(s',p') = (2\pi)^d \delta_{s,s'} \delta^d(p-p'). \tag{3.72}$$

Then when our theory includes interactions, $a_{s,p}$ generally exhibits a dependence on z, which we have explicitly included in Eq. (3.71).

Under the normalization Eq. (3.72), the a algebra satisfies

$$[a_{s,p}(z), a^{\dagger}_{s',p'}(z)] = (2\pi)^d \delta_{s,s'} \delta^d(p-p'). \tag{3.73}$$

Since the $\varphi^{(l)}$'s transform as simple representations of the AdS isometries, we are in a good position to consider the ET.

3.5.1 AdS ET via Analysis of Wave Functions

The ET is largely a statement about the degrees of freedom contained in a spin-l field in its massless limit. For a representation V of a group G to qualify as irreducible,

[4]Previously, we foliated in time. The definition of the inner product remains the same as that presented in appendix §D with the replacement $t \to z$.

the following must hold:

$$Gv = V, \qquad \forall\, v \in V. \tag{3.74}$$

In flat space, the transversality constraint for a massive field implies all polarizations must be space-like. In other words, one can go to a frame in which there are only non-vanishing components for spatial indices. In the massless limit, polarizations must be either space-like or light-like. In particular, some linear combinations of the polarizations must now be light-like. Light-like objects can only be boosted to other light-like objects, thus pulling the particular polarizations out of the spin-l orbit into a spin-$(l-1)$ orbit: $V_l \to V_l' \oplus V_{l-1}$, where V_l' still transforms as a spin-l representation, but is a smaller dimension than V_l (it is spanned by only space-like polarizations). In practice, this means for some s that $\varphi^{(l)}_{\mu_1 \ldots \mu_l}(s,p) = \partial_{(\mu_1}\varphi^{(l-1)}_{\mu_2 \ldots \mu_l)}(p)$ in flat space. In AdS, this means we wish to find all s such that

$$\begin{aligned}
\varphi^{(l)}_{M_1 \ldots M_l}(s,p) &\underset{\frac{\Delta_J}{|p|} \ll 1}{=} \nabla_{(M_1}\varphi^{(l-1)}_{M_2 \ldots M_l)}(p) \\
&\underset{\frac{\Delta_J}{|p|} \ll 1}{=} \partial_{(M_1}\varphi^{(l-1)}_{M_2 \ldots M_l)}(p),
\end{aligned} \tag{3.75}$$

where $(\ldots)$ denotes complete symmetrization of the indices. The second line follows since the large $|p|$ limit, $\frac{\Delta_J}{|p|} \ll 1$, implies derivative terms dominate and the contribution of the Christoffel symbol is negligible. The $(l-1)$ mode function, $\varphi^{(l-1)}$, may itself correspond to several potential polarizations, so we have foregone labeling it.

Now if $\varphi^{(l)}$ and $\nabla\varphi^{(l-1)}$ satisfy the same equations, then Eq. (3.75) holds. In the large $|p|$ limit, we expect terms in Eq. (3.70) that go as $\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and ∂_z^2

to dominate; we additionally wish to maintain proper boundary asymptotics, so we insist on keeping terms that go as ∂_z as well. All terms that do not involve a derivative are to be ignored. Consequently, Eq. (3.70) becomes

$$\nabla^{2\,N_1\ldots N_l}_{M_1\ldots M_l}\varphi^{(l)}_{N_1\ldots N_l} = 0. \tag{3.76}$$

Acting on $\varphi^{(l)}$ with $g^{AB}\nabla_A\nabla_B$ and keeping only derivative terms yields

$$\begin{aligned} \nabla^2_{scalar}\varphi^{(l)}_{M_1\ldots M_l} - 2g^{AB}\Gamma^C_{A(M_1}\partial_B\varphi^{(l)}_{C\ldots M_l)} =& 0 \implies \\ \left[-z^2\partial_z^2 + (d-1)z\partial_z + z^2\partial^2\right]\varphi^{(l)}_{M_1\ldots M_l} - 2g^{AB}\Gamma^C_{A(M_1}\partial_B\varphi^{(l)}_{C\ldots M_l)} =& 0, \end{aligned} \tag{3.77}$$

where the symmetrizer $(\ldots)$ acts on the M indices only. To be explicit about the difference between the scalar equation of motion and the spin-l equations, note that each index contributes another term of the form $\Gamma\partial\varphi^{(l)}$. Explicitly, this operator is

$$\Gamma^C_{AM}g^{AB}\partial_B = z\left[\delta^C_M\partial_z - \delta^C_z\partial_M\right], \tag{3.78}$$

where the term $-\delta^z_M\eta^{BC}\partial_B$ since its action is trivial on transverse fields in this limit. The first term in Eq. (3.78) simply differentiates $\varphi^{(l)}$ with respect to z and adds l of such terms. This can be assimilated into the operator in brackets in Eq. (3.77) to send $(d-1) \to (d-2l-1)$. The second term in Eq. (3.78) simply results in the unsurprising symmetrized derivative term $z\partial_{(M_1}\varphi^{(l)}_{z\ldots M_l)}$. Revisiting Eq. (3.77) under this prescription yields

$$\left[-z^2\partial_z^2 + (d-2l-1)z\partial_z + z^2\partial^2\right]\varphi^{(l)}_{M_1\ldots M_l} + 2z\partial_{(M_1}\varphi^{(l)}_{z\ldots M_l)} = 0. \tag{3.79}$$

In flat space, the light-like polarization in the massless limit is special because its orbit under the Lorentz group is just other light-like polarizations. Recall in the spin-1 case in AdS that mode functions with a non-vanishing z-component could not be transformed to mode functions with vanishing z-component when acted upon by the AdS isometry group in the massless limit. It was, indeed, the analogue to the light-like longitudinal polarization as a consequence. We keep this fact in mind and note the special appearance of $\varphi^{(l)}_{z\ldots M_l}$ in Eq. (3.79). Proceeding, we choose our mode functions such that they divide naturally into those with no z-component (for a single index) and those with only the z-component as the degree of freedom (with non-z-components appearing as derivatives of the z-component) and consider the equations of motion for $M_1 = z$:

$$\left[-z^2\partial_z^2 + (d-2(l-1)-1)z\partial_z + z^2\partial^2\right]\varphi^{(l)}_{zM_2\ldots M_l} + 2z\partial_{(M_2}\varphi^{(l)}_{zz\ldots M_l)} = 0. \tag{3.80}$$

This is precisely the equations $\varphi^{(l-1)}$ satisfies. Differentiating Eq. (3.80) with respect to x^{M_1}, symmetrizing, and throwing out terms without derivatives yields

$$\left[-z^2\partial_z^2 + (d-2l-1)z\partial_z + z^2\partial^2\right]\partial_{(M_1}\varphi^{(l-1)}_{M_2\ldots M_l)} + 2z\partial_{(M_1}\partial_{(M_2}\varphi^{(l-1)}_{\ldots M_l))} = 0. \tag{3.81}$$

This is exactly Eq. (3.79). We thus conclude that mode functions whose only degrees of freedom come from setting one of its indices to z are exactly the mode functions we seek in the ET. That is, mode functions such that

$$\varphi^{(l)}(s=z,p)_{M_1\ldots M_l} = \begin{cases} \varphi^{(l)}(s=z,p)_{z\ldots M_l} & M_1 = z \\ \partial_{(\mu}\partial^{-2}\partial_z\varphi^{(l)}(s=z,p)_{z\ldots M_l)} & M_1 = \mu \end{cases}. \tag{3.82}$$

We may thus write our original AdS field in the "high-energy" limit as

$$\phi_{M_1\ldots M_l} \to \phi^0_{M_1\ldots M_l} + \nabla_{(M_1}\theta_{M_2\ldots M_l)} \tag{3.83}$$

for some spin-$(l-1)$ field $\theta_{M_1\ldots M_{l-1}}$.

If the trace and non-transverse components were left as gauge degrees of freedom, the above would describe a theory for which the Lagrangian was of the form $\mathcal{L}(\phi_{M_1\ldots M_l} - \nabla_{(M_1}\theta_{M_2\ldots M_l)})$. The spontaneously broken gauge transformations are given by $\delta\phi_{M_1\ldots M_l} = \nabla_{(M_1}\epsilon_{M_2\ldots M_l)}$ while the Goldstone mode θ is simply shifted by ϵ.

It is worthwhile to remark that we could have repeated this process of setting an index to z in Eq. (3.79) to obtain a tower in which we ultimately conclude

$$\varphi^{(l)}_{\text{scalar}}(p)_{M_1\ldots M_l} = \partial_{(M_1}\ldots\partial_{M_l)}\varphi(p) \tag{3.84}$$

for a scalar mode, φ. This mode function contributes the most in the high-energy limit, and thus, unsurprisingly, we can write the dominant contribution to the theory as

$$\phi_{M_1\ldots M_l} \to \nabla_{(M_1}\ldots\nabla_{M_l)}\theta. \tag{3.85}$$

3.5.2 Spin-l AdS ET at the Boundary

As in the spin-1 case, a free theory trivially satisfies the ET. If the interactions that are added are only excited at high-energies (short distances), corresponding to heavy exchange operators, $\Delta \gg \Delta_J$, then the equivalence theorem holds in position space when all scales of $|p|$ are integrated over. At the boundary, the equivalence theorem then takes the form

$$\begin{aligned}\partial^\mu \mathcal{O}_{\mu\mu_1\ldots\mu_{l-1}} =& \lim_{z\to 0} \frac{1}{z^{\delta-l}} z^{d-1}\partial_z \left[\frac{1}{z^{d-1}}\phi_{z\mu_1\ldots\mu_{l-1}}\right] \\ =& \lim_{z\to 0} \frac{1}{z^{\delta-(l-1)}}\phi_{\mu_1\ldots\mu_{l-1}} \\ =& \mathcal{O}_{\mu_1\ldots\mu_{l-1}},\end{aligned} \tag{3.86}$$

which is to say the spin-l conformal current is not conserved and its divergence is approximately a primary spin-$(l-1)$ current when computing correlators. Continuing with the tower prescription discussed in the previous section, we may write

$$\mathcal{O}_{\mu_1\ldots\mu_l} = \mathcal{O}^0_{\mu_1\ldots\mu_l} + \partial_{(\mu_l}\partial^{-2}\mathcal{O}^0_{\mu_1\ldots\mu_{l-1})} + \cdots + \partial_{\mu_1}\ldots\partial_{\mu_l}\left(\partial^{-2}\right)^l\mathcal{O}, \tag{3.87}$$

where each $\mathcal{O}^0$ is a conserved primary current.

Of course, this allows one to write approximately

$$\mathcal{O}_{\mu_1\ldots\mu_l} \approx \partial_{\mu_1}\ldots\partial_{\mu_l}\left(\partial^{-2}\right)^l\mathcal{O}, \tag{3.88}$$

thus reducing the problem of computing spin-l conformal correlators to computing computing a scalar correlator.

3.5.3 Generalization to CFT ET

That the divergence of a non-conserved conformal current is approximately primary in theories with particular bulk couplings is the most interesting consequence of the AdS ET. While bulk gauge theories are always dual to theories with conformal currents, the inverse mapping is not unique, and it is of interest to confirm that the divergence of conformal currents is generally primary in particular limits under different bulk theories. It would also be useful to examine any additional consequences or constraints of the ET for conformal theories. We thus seek a purely conformal, bottom-up approach to the ET.

The wave functions that played such a central role in the purely AdS approach to the ET are irreducible representations of the conformal group. The matrix elements, S, are built out of products of these functions and, consequently, can be expressed as sums over other irreducible representations of the conformal group. This is akin to the procedure of expanding conformal correlators in irreducible representations of the conformal group as conformal blocks. Since we may interpret the matrix elements as Fourier transforms of conformal correlators, and the AdS ET arises from a relationship between scalar and vector wave functions, it seems natural to examine the ET on the CFT side as a relationship between conformal blocks of currents and of scalars. In the following sections, we thus analyze the CFT ET using the machinery of the conformal block expansion.

3.6 Equivalence Theorem in CFTs

In this section, we will generalize the ET in terms of conformal blocks. We begin with a warm-up example in two dimensions, which will serve as both a simple introduction and a distinct contrast to the more interesting four dimensional case. We emphasize that the $d = 2$ example is merely an explication of ideas that are known in the literature (see, e.g. [82–85]). It is well known that in $d = 2$ the conformal group naturally breaks up into holomorphic/antiholomorphic (also called 'left moving/right moving') parts. Consequently, the conformal blocks themselves factorize into holomorphic and antiholomorphic terms, which greatly facilitates the analysis of spinning correlators. We will then move onto the more involved $d = 4$ case, where we will review the index-free formalism of [1] and [2] and determine the ET as a statement of a spin-1 CFT current.

3.6.1 Two Dimensional Warm-Up

A hallmark feature of 2-d CFTs is that one may change coordinates to the complex plane by noting that the line element may be expressed as $ds^2 = dx^2 + dy^2 = dzd\bar{z}$, where $z = x + iy$. Therefore, we can write the metric as

$$g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \tag{3.89}$$

where a and b denote z or $\bar{z}$. Under this coordinate change, the current J^μ can be mapped to a two component vector

$$\begin{pmatrix} J_z \\ J_{\bar{z}} \end{pmatrix} = \begin{pmatrix} J_x + iJ_y \\ J_x - iJ_y \end{pmatrix}. \tag{3.90}$$

If one is interested in correlators that involve only J and hermitian operators, then it is clear that one may consider only one component and obtain the other by complex conjugation. In general, J may be a higher spin current, in which case one must consider the total independent degrees of freedom classified by their weights $(h, \bar{h})$. We will elaborate on this point when we examine the spin-3 current.

Unitarity bounds of CFT currents, which are typically calculated by bounding the norm of descendant states, have clear bulk interpretations. If the scaling dimension of the spin-ℓ current exceeds $d - 2 + \ell$, then this corresponds to a massive bulk gauge boson. Conversely, when the bound is saturated, the current must be dual to massless bulk gauge boson and $\partial \cdot J = 0$ so that no degrees of freedom are lost or gained. It is evident then that any statement of the equivalence theorem in a traditional CFT must involve only those currents that do not saturate the bound, so that there are longitudinal bulk propagating degrees of freedom. So before we move onto the ET, it will be worthwhile to clarify how these bounds show up in the divergences of correlation functions. In two dimensions, it is possible to check this bound by considering the divergence around a small ball Ω

$$\int_\Omega d^2x \partial_\mu K^\mu, \tag{3.91}$$

where K^μ may be any correlator of J^μ. We can write this as a surface integral

$$\begin{aligned}\int_\Omega d^2x\partial_\mu K^\mu &= \int_{\partial\Omega} dA_\mu K^\mu \\ &= \frac{i}{2}\int_{\partial\Omega}\left(d\bar{z}K^z - dzK^{\bar{z}}\right), \end{aligned} \tag{3.92}$$

where dA_μ runs along the counterclockwise contour. For a given correlator involving an arbitrary number of operators, the above integrals are difficult to evaluate in full generality. However, we may consider a simple example of a three point function involving only a single spin-1 current

$$\langle J^z(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\rangle = \frac{1}{|z|^{2h_{23}+\Delta_J}|z-1|^{2h_{32}+\Delta_J}}\left[\frac{\bar{z}(\bar{z}-1)}{z(z-1)}\right]^{1/2}, \tag{3.93}$$

where $h_{ij} \equiv h_i - h_j$ and we have mapped $z_1 \to z$, $z_2 \to 0$, $z_3 \to 1$. The $\bar{z}$ component is identical, except the exponent of the square bracket term is $-1/2$. It is easy to check that Eq. (3.92) vanishes up to contact terms when $\Delta = \ell = 1$ and when the scaling dimensions of the two scalars are the same. Although all of the above discussion is rather obvious, it is worth mentioning because the expressions we will soon encounter (which are divergences of correlators) may not appear to be zero up to contact terms at first glance when $\Delta_J = \ell$, but internal consistency can be checked using the above method.

3.6.1.1 Spin-1 Current

Consider, then, the four-point function

$$G_4 \equiv \langle J_1(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\mathcal{O}(z_3,\bar{z}_3)\mathcal{O}(z_4,\bar{z}_4)\rangle, \tag{3.94}$$

where we will denote $J^z \equiv J$ and $J^{\bar{z}} \equiv \bar{J}$. For computational simplicity, we will take all three scalars to have the same scaling dimensions so

$$h_{\mathcal{O}_i} = \bar{h}_{\mathcal{O}_i} \equiv h = \frac{\Delta}{2}, \qquad (i = 1, 2, 3). \tag{3.95}$$

It is important to distinguish the spin of AdS_3 gauge boson and the spin of the current. While it is true that J^μ is a spin-1 current in that it has one index, the reducibility of the conformal group lets us classify correlator purely by the *components* of J^a, as long as J is primary[5]. Here, the two spin states, classified by $(h, \bar{h}) = (1, 0)$ and $(0, 1)$, correspond to J and $\bar{J}$, respectively. The correlation function can be written as an overall scale term times a general function of the conformally invariant cross ratio $\eta \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}$:

$$G_4^z = \frac{1}{|z_{12}|^{\Delta_j+\Delta}} \frac{1}{|z_{34}|^{2\Delta}} \left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta} \left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{1/2} f_1(\eta, \bar{\eta}) \tag{3.97}$$

$$G_4^{\bar{z}} = \frac{1}{|z_{12}|^{\Delta_j+\Delta}} \frac{1}{|z_{34}|^{2\Delta}} \left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta} \left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{-1/2} f_2(\eta, \bar{\eta}) \tag{3.98}$$

How should one deal with the functions f_1 and f_2? The four point function may be regarded as gluing together three point functions via the insertion of states corre-

[5]Otherwise a component of a tensor with a z indices and b $\bar{z}$ indices would not transform as

$$J^{z_1 z_2 \ldots z_a \bar{z}_1 \bar{z}_2 \ldots \bar{z}_b}(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^a \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^b J^{z_1 z_2 \ldots z_a \bar{z}_1 \bar{z}_2 \ldots \bar{z}_b}(z, \bar{z}), \tag{3.96}$$

with $z \to f(z)$.

sponding to exchanged operators of weights $(h_e, \bar{h}_e)$[6]:

$$G_4 = \sum_{h_e,\bar{h}_e} \langle J_1\mathcal{O}|h_e,\bar{h}_e\rangle\langle h_e,\bar{h}_e|\mathcal{O}\mathcal{O}\rangle. \tag{3.100}$$

One can then package the contribution of a given primary and descendants to the four point function in the form of "conformal blocks". That is, if the f_i admit the following expansion:

$$f_i(z,\bar{z}) \sim \sum_{h_e,\bar{h}_e} \lambda_{h_e,\bar{h}_e}\left[\mathcal{L}_{h_e,\bar{h}_e}(z,\bar{z}) + \mathcal{R}_{\bar{h}_e,h_e}(z,\bar{z})\right], \tag{3.101}$$

then $\mathcal{L}_{h_e,\bar{h}_e}$ represents the contribution of a given primary and its descendants to the four point function and the second term $\mathcal{R}_{\bar{h}_e,h_e}$ represents the contribution of its conjugate, and the coefficients $\lambda_{h_e,\bar{h}_e}$ characterize the dynamics of the bulk theory. It is important to emphasize that the sum above runs only over the weights of exchanged primaries. The contributions of a single primary and its descendants have been summed into the blocks $\mathcal{L}$ and $\mathcal{R}$, whereas in Eq. (3.99), the sum runs over the weights of all exchanged operators. If all the exchanged operators correspond to scalar primaries and descendants (as we have assumed thus far), then $\mathcal{R}$ should be obtainable by simply swapping the arguments of $\mathcal{L}$. Consistent with the theme of left/right classification in two dimensional CFTs, we see that then the expansion

[6] Note here that the divergence of the correlator is given by

$$\begin{aligned}\partial_\mu G_4^\mu = \bar{\partial}G_4^z + \partial G_4^{\bar{z}} &= \sum_{h_e,\bar{h}_e}\left[\bar{\partial}\langle J\mathcal{O}|h_e,\bar{h}_e\rangle\langle h_e,\bar{h}_e|\mathcal{O}\mathcal{O}\rangle + \partial\langle\bar{J}\mathcal{O}|h_e,\bar{h}_e\rangle\langle h_e,\bar{h}_e|\mathcal{O}\mathcal{O}\rangle\right]\\ &= \sum_{h_e,\bar{h}_e}\left[\bar{\partial}\langle J\mathcal{O}\mathcal{O}_{h_e,\bar{h}_e}\rangle + \partial\langle\bar{J}\mathcal{O}\mathcal{O}_{h_e,\bar{h}_e}\rangle\right]\langle h_e,\bar{h}_e|\mathcal{O}\mathcal{O}\rangle. \end{aligned} \tag{3.99}$$

The term in the square brackets is $\partial_\mu\langle J^\mu\mathcal{O}\mathcal{O}_{h_e,\bar{h}_e}\rangle$. Therefore, if the three point functions involving J and $\bar{J}$ satisfy the Ward identity, it is evident that the four point function will as well.

would factorize in a trivial way.

Dolan and Osborn [86] determined the functions $\mathcal{L}_{h_e,\bar{h}_e}$ and $\mathcal{R}_{\bar{h}_e,h_e}$ in terms of hypergeometric functions when the exchanged operators were symmetric traceless primaries. As foreseen, these functions factorize into left and right parts

$$\begin{aligned}\mathcal{L}_{h_e,\bar{h}_e} = k_{2h_e}(\eta)k_{2\bar{h}_e}(\bar{\eta}) = \eta^{h_e}\,{}_2F_1(h_e - h_j + h, h_e; 2h_e; \eta)\\ \times\,\bar{\eta}^{\bar{h}_e}\,{}_2F_1(\bar{h}_e - h_j + h + 1, \bar{h}_e; 2\bar{h}_e; \bar{\eta}),\end{aligned} \tag{3.102}$$

and

$$\mathcal{R}_{\bar{h}_e,h_e} = \mathcal{L}_{h_e,\bar{h}_e}(\eta \leftrightarrow \bar{\eta}). \tag{3.103}$$

Therefore, the four point function is given by

$$\begin{aligned}G_4^a \sim & S^a \sum_{h_e,\bar{h}_e} \lambda^a_{h_e,\bar{h}_e} \Big(\eta^{h_e}\,{}_2F_1(h_e - h_j + h, h_e; 2h_e; \eta)\\ & \times\,\bar{\eta}^{\bar{h}_e}\,{}_2F_1(\bar{h}_e - h_j + h + 1, \bar{h}_e; 2\bar{h}_e; \bar{\eta}) + \eta \leftrightarrow \bar{\eta}\Big)\\ \equiv & S^a \sum_{h_e,\bar{h}_e} \lambda^a_{h_e,\bar{h}_e} \left[k_{2h_e}(\eta)k_{2\bar{h}_e}(\bar{\eta}) + k_{2h_e}(\bar{\eta})k_{2\bar{h}_e}(\eta)\right],\end{aligned} \tag{3.104}$$

where we have encapsulated the scale term (the prefactors of the f_i) as S^a (or S and $\bar{S}$ for brevity). It is important to mention that since there are two degrees of freedom corresponding to the two helicity states of the bulk gauge boson, f_1 and f_2 (which we will call f and $\bar{f}$ since they are associated with the z and $\bar{z}$ components, respectively) will have independent expansion coefficients. It is clear that the divergence $\bar{\partial}G + \partial\bar{G}$ will clearly result in the divergence of the scale terms $(\bar{\partial}Sf + \partial\bar{S}\bar{f})$ plus the divergence of the blocks themselves ($S\bar{\partial} f + \bar{S}\partial \bar{f}$). It is the latter that will be of importance to us since the divergence of the scale term will result in another scale term, but the blocks themselves will remain unchanged. Let us consider the term $\bar{S}\partial \bar{f}$. Apart from the fact that the OPE coefficients are different, there is no difference between f and $\bar{f}$, so the analysis that will follow applies to the term $S\bar{\partial} f$ as well. Lastly, the blocks are functions of the anharmonic ratio and its conjugate, η and $\bar{\eta}$ while the derivative acts with respect to z or $\bar{z}$. While it is true that we may fix coordinates such that $\eta \to z$ and $\bar{\eta} \to \bar{z}$, there is no unique map that does this without causing the scale term to diverge. We must then take $\partial = \partial\eta \frac{\partial}{\partial \eta}$ and $\bar{\partial} = \bar{\partial}\bar{\eta}\frac{\partial}{\partial \bar{\eta}}$. With this out of the way, we have

$$\begin{aligned}
\partial \bar{f} = \sum_{h_e, \bar{h}_e} \lambda^{\bar{z}}_{h_e, \bar{h}_e} \Bigg[\partial\eta \Bigg(& h_e \eta^{h_e - 1} {}_2F_1(h_e - h_j + h, h_e; 2h_e; \eta) \\
& + \frac{1}{2}(h + h_e - h_j)\eta^{h_e} {}_2F_1(1 + h + h_e - h_j, h_e + 1, ; 1 + 2h_e; \eta) \Bigg) k_{2\bar{h}_e}(\bar{\eta}) \\
& + \partial\eta \Bigg(\bar{h}_e \eta^{\bar{h}_e - 1} {}_2F_1(\bar{h}_e - h_j + h + 1, \bar{h}_e; 2\bar{h}_e; \eta) \\
& + \frac{1}{2}(h + \bar{h}_e - h_j + 1) {}_2F_1(2 + h + \bar{h}_e - h_j, \bar{h}_e + 1; 1 + 2\bar{h}_e; \eta) \Bigg) k_{2h_e}(\bar{\eta}) \Bigg]. \qquad (3.105)
\end{aligned}$$

Already at this level, we can see a very simple version of the ET. Suppose h_e is large compared to both h_j, h, and is comparable to $\bar{h}_e$. The second condition means that $\bar{h}_e = h_e - \ell \approx h_e$ i.e. the twists of the exchanged operators are dominated by their scaling dimensions[7]. Then, the above result readily factorizes into scalar blocks. This

[7] More precisely, the exchanged operators may be classified by their irreducible components, given by weights $(h_e, h_e - \ell)$, $(h_e - 1, h_e - \ell + 1)$, etc. down to $(h_e - \ell, h_e)$. If the twists of these components are not dominated by ℓ, i.e. $\ell \ll h_e$, then we may assume that $\bar{h}_e \sim h_e$ for all states in the $h_e \to \infty$ limit.

can be seen by the following trivial expansions:

$$\eta^{h_e-1} \equiv \eta^{h_e(1-\epsilon)} \approx \eta^{h_e} + \mathcal{O}(\epsilon); \tag{3.106}$$

$$\begin{aligned}
&{}_2F_1(1+h+h_e-h_j, h_e+1; 1+2h_e; \eta) \approx {}_2F_1(h_e(1+\epsilon), h_e(1+\epsilon); (\epsilon+2)h_e; \eta) \\
&= 1 + \frac{(1+\epsilon)^2 h_e \eta}{2+\epsilon} + \frac{(1+\epsilon)^2 h_e (1+h_e(1+\epsilon))^2 \eta^2}{2(2+\epsilon)(1+h_e(2+\epsilon))} + \ldots \\
&= {}_2F_1(h_e, h_e; 2h_e; \eta) + \mathcal{O}(\epsilon).
\end{aligned} \tag{3.107}$$

Plugging these relations, in the $h_e \to \infty$ limit, we see that

$$\begin{aligned}
\partial \bar{f} \to \sum_{h_e, \bar{h}_e} \lambda^{\bar{z}}_{h_e, \bar{h}_e} \partial\eta \Bigg[&\left(h_e + \frac{1}{2}(h + h_e - h_j)\right) k_{2h_e}(\eta) k_{2\bar{h}_e}(\bar{\eta}) \\
&+ \left(\bar{h}_e + \frac{1}{2}(h + \bar{h}_e - h_j + 1)\right) k_{2\bar{h}_e}(\eta) k_{2h_e}(\bar{\eta}) \Bigg] \\
\approx \sum_{h_e, \bar{h}_e} \lambda^{\bar{z}}_{h_e, \bar{h}_e} \partial\eta &\left(h_e + \frac{1}{2}(h + h_e - h_j)\right) \left[k_{2h_e}(\eta) k_{2\bar{h}_e}(\bar{\eta}) + k_{2\bar{h}_e}(\eta) k_{2h_e}(\bar{\eta})\right],
\end{aligned} \tag{3.108}$$

where again we have assumed that $\bar{h}_e \sim h_e$. We immediately see in the above equation the appearance of the scalar blocks of the form $k_i(\eta)k_j(\bar{\eta}) + k_j(\eta)k_i(\bar{\eta})$, weighted by some coefficient that depends on the twist of the exchanged operator. Here, we see that the divergence leads to a term that is at most linear in the twist whereas in the four dimensional case, we will see that there exists a tower of operators beginning with terms proportional to Δ^{s+1}, where s is the spin of the current, down to $\mathcal{O}(1)$ terms. By contrast, we see that in two dimensions, all terms contribute equally in the power of the exchanged weights. It should also be evident that the other terms in the divergence do not spoil the analysis above (we will compute all such terms in the four dimensional case and show this in full generality, as this is just a pedagogical example). The divergence of the scale terms will result in other scale terms, but they remain prefactors of scalar blocks. The remaining term $\bar{\partial} f$ will result in Eq. (3.108) with the replacement $\lambda^{\bar{z}}_{h_e,\bar{h}_e} \to \lambda^{z}_{h_e,\bar{h}_e}$ and $\partial\eta \to \bar{\partial}\bar{\eta}$.

3.6.1.2 Spin-3 Current

Not surprisingly, higher spin correlators in two dimensional CFTs are as easy to handle as the spin-1 case and the ET for these objects essentially reduces to the analysis of last section. Consider

$$G_4 \equiv \langle S(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle, \tag{3.109}$$

where S is a spin-3 operator. We must clarify how many independent degrees of freedom there are. After the usual change of coordinates, where $S^{\mu\nu\sigma} \to S^{abc}$, we may eliminate 4 degrees of freedom assuming S is symmetric and therefore components of S are discriminated only by the number of z and $\bar{z}$ indices[8]. They are

$$S^{zzz}, \qquad S^{zz\bar{z}}, \qquad S^{z\bar{z}\bar{z}}, \qquad S^{\bar{z}\bar{z}\bar{z}}. \tag{3.110}$$

The idea is each of the above components are themselves irreducible components of the global conformal group and we must compute the associated four point function for each of them. Furthermore, we would like to emphasize that since these really are independent components with different weights, we will have four distinct functions

[8] Unitarity places bounds on the weights $(h, \bar{h})$. However, there is no loss in generality if we relax these bounds.

of η and $\bar{\eta}$, which we will label by

$$f_1(\eta,\bar{\eta}), \qquad f_2(\eta,\bar{\eta}), \qquad f_3(\eta,\bar{\eta}), \qquad f_4(\eta,\bar{\eta}), \tag{3.111}$$

analogous to the spin-1 case. The spin-3 current may then be regarded as four independent expansions in conformal blocks. The following expressions for the components, labeled by the operator weights, are then evident:

$$(3,0): \qquad G_4^{zzz} = \frac{1}{|z_{12}|^{\Delta_j+\Delta}}\frac{1}{|z_{34}|^{2\Delta}}\left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta}\left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{3/2} f_1(\eta,\bar{\eta}) \tag{3.112}$$

$$(2,1): \qquad G_4^{zz\bar{z}} = \frac{1}{|z_{12}|^{\Delta_j+\Delta}}\frac{1}{|z_{34}|^{2\Delta}}\left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta}\left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{1/2} f_2(\eta,\bar{\eta}) \tag{3.113}$$

$$(1,2): \qquad G_4^{z\bar{z}\bar{z}} = \frac{1}{|z_{12}|^{\Delta_j+\Delta}}\frac{1}{|z_{34}|^{2\Delta}}\left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta}\left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{-1/2} f_3(\eta,\bar{\eta}) \tag{3.114}$$

$$(0,3): \qquad G_4^{\bar{z}\bar{z}\bar{z}} = \frac{1}{|z_{12}|^{\Delta_j+\Delta}}\frac{1}{|z_{34}|^{2\Delta}}\left(\frac{|z_{24}|}{|z_{14}|}\right)^{\Delta_j-\Delta}\left(\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}}\right)^{-3/2} f_4(\eta,\bar{\eta}). \tag{3.115}$$

The divergence can then be expressed as the two by two matrix

$$\partial_\mu G_4^{\mu\nu\sigma} \to \begin{pmatrix} \bar{\partial} G_4^{zzz} + \partial G_4^{\bar{z}zz} & \bar{\partial} G_4^{zz\bar{z}} + \partial G_4^{\bar{z}z\bar{z}} \\ \bar{\partial} G_4^{z\bar{z}z} + \partial G_4^{\bar{z}\bar{z}z} & \bar{\partial} G_4^{z\bar{z}\bar{z}} + \partial G_4^{\bar{z}\bar{z}\bar{z}} \end{pmatrix}. \tag{3.116}$$

Before we go any further, let us take stock of some recurring themes. Four point functions of any spin current with three other scalars can be characterized completely by the exponent of the $\frac{\bar{z}_{12}z_{24}\bar{z}_{14}}{z_{12}\bar{z}_{24}z_{14}} \equiv \beta$ term. If we denote the common prefactor in the above Eqs. (3.112) - (3.115) as α, then they reduce simply to the form $\alpha\beta^k f_i(\eta,\bar{\eta})$. For a spin-3 current, have 3+1 degrees of freedom and so the irreducible states are distinguishable only by their values of k given above: $3/2$, $1/2$, $-1/2$, and $-3/2$ (corresponding to $s = 3,\ 1,\ -1$, and -3) along with their partial wave expansion coefficients. The general pattern we see is that components of symmetric four point functions involving a higher spin current $\langle J^s \mathcal{O}\mathcal{O}\mathcal{O}\rangle$ are distinguishable by $k = \frac{s}{2}, \frac{s-2}{2}, \ldots, -\frac{s}{2}$, corresponding to the $zzz\ldots z$, $zzz\ldots\bar{z}$, $\ldots$, $\bar{z}\bar{z}\bar{z}\ldots\bar{z}$ components and each independent component will be associated with a different function of η and $\bar{\eta}$ (that is, we have as many independent functions as independent components). Therefore, it is simple to extend the ET to hold for any higher spin correlator. With all this in mind, Eq. (3.116) can be written as a "scale" matrix (which includes the divergence of the scale terms) plus the derivatives acting on the f_i:

$$\begin{pmatrix} f_1\partial_{111} + f_2\partial_{112} & f_1\partial_{121} + f_3\partial_{122} \\ f_1\partial_{211} + f_3\partial_{212} & f_2\partial_{221} + f_4\partial_{222} \end{pmatrix} + \alpha \begin{pmatrix} \beta^{1/2}\partial f_2 + \beta^{3/2}\bar{\partial} f_1 & \beta^{1/2}\bar{\partial} f_1 + \beta^{-1/2}\partial f_3 \\ \beta^{1/2}\bar{\partial} f_1 + \beta^{-1/2}\partial f_3 & \beta^{-1/2}\bar{\partial} f_3 + \beta^{-3/2}\partial f_4 \end{pmatrix}. \tag{3.117}$$

$$\partial_{111} = \left(\bar{\partial}\alpha\beta^{3/2} + \frac{3\alpha\beta^{1/2}\bar{\partial}\beta}{2}\right) \tag{3.118}$$

$$\partial_{112} = \left(\partial\alpha\beta^{1/2} + \frac{\alpha\beta^{-1/2}\partial\beta}{2}\right) \tag{3.119}$$

$$\partial_{121} = \left(\bar{\partial}\alpha + \frac{\alpha\beta^{-1/2}\bar{\partial}\beta}{2}\right) \tag{3.120}$$

$$\partial_{122} = \left(\partial\alpha\beta^{-1/2} - \frac{\beta^{-3/2}\alpha\partial\beta}{2}\right) \tag{3.121}$$

$$\partial_{211} = \left(\bar{\partial}\alpha + \frac{\alpha\beta^{-1/2}\bar{\partial}\beta}{2}\right) \tag{3.122}$$

$$\partial_{212} = \left(\partial\alpha\beta^{-1/2} - \frac{\beta^{-3/2}\alpha\partial\beta}{2}\right) \tag{3.123}$$

$$\partial_{221} = \left(\bar{\partial}\alpha\beta^{-1/2} - \frac{\beta^{-3/2}\alpha\bar{\partial}\beta}{2}\right) \tag{3.124}$$

$$\partial_{222} = \left(\partial\alpha\beta^{-3/2} - \frac{3\alpha\beta^{-5/2}\partial\beta}{2}\right) \tag{3.125}$$

Clearly, the first term leaves the conformal blocks unchanged. Combined with the result of the last section,

$$\partial f_i \approx \sum_{h_e,\bar{h}_e} \lambda^i_{h_e,\bar{h}_e} \partial\eta \left(h_e + \frac{1}{2}(h + h_e - h_j)\right) \left[k_{2h_e}(\eta)k_{2\bar{h}_e}(\bar{\eta}) + k_{2\bar{h}_e}(\eta)k_{2h_e}(\bar{\eta})\right], \tag{3.126}$$

we see that the second term of Eq. (3.117) also reduces to a matrix whose components are scalar blocks in the high energy limit of exchanged operator weights.

3.6.2 Higher Dimensions

We will now move onto the more interesting case of the ET in $d = 4$. The four dimensional case is qualitatively different from the two dimensional case (i.e. the conformal group does not factorize) and analysis of correlation functions involving spinning fields is often a tremendous computational exercise. The required computations in position space are (in principle) tractable but in practice, the propagation of indices and counting of possible tensor structures makes for a difficult calculation.

Recently, the authors of [1] developed an index-free formalism where the index of a current J is encoded into auxiliary vectors z_{μ_i}. Great simplifications occur when one lifts the index-free correlator to embedding space, where the x's project to P's and the z's project to Z's. Namely, we find that terms which are $\mathcal{O}(P_i \cdot Z_i)$ and $\mathcal{O}(P^2, Z^2)$ are redundant and so we do not need to include them in the calculation. Although this method facilitates a great deal of intermediate calculations, we will have to project back down to physical space at the end of the day in order to consider the conservation operator. For completeness, we will review this formalism below, but readers familiar with these concepts can skip to §3.6.2.2.

3.6.2.1 Review of Index-Free Formalism of [1] and [2]

The index-free formalism requires some familiarity with the embedding space or null cone approach to CFTs [87]. The idea of embedding space — one that hearkens back to Dirac [88] — is now pervasive in the CFT literature. For this reason, we will omit reviewing this subject in great detail and instead direct readers who are unfamiliar with these ideas to the pedagogical review in [89]. The discussion below is meant to bring the reader up to speed with the material needed for the $d = 4$ calculation as quickly as possible, so the interested reader is also encouraged to consult the original literature vis-á-vis the index-free approach.

The essential idea of [1] is to encode the spin ℓ of a (symmetric) field by contracting it with vectors $z_{\mu_1} z_{\mu_2} \ldots z_{\mu_\ell}$:

$$f(z) \equiv f_{\mu_1,\ldots,\mu_\ell} z^{\mu_1 \cdots} z^{\mu_\ell}. \tag{3.127}$$

If the field has the added bonus of being traceless, then it may be recovered from $f(z)$ by restricting the polynomial to the region where $z^2 = 0$. This is seen from the fact that tracelessness implies $f(z)$ will be harmonic[9] and any polynomial may be written in the form of $h(z) + z^2 j(z)$, where $h(z)$ is harmonic, so $f(z)|_{z^2=0} = h(z)$. Another way of looking at this would be to note that a symmetric traceless tensor differs from a purely symmetric one by terms of $\mathcal{O}(z^2)$.

One can go further by lifting tensors to embedding space $f^{\mu_1,\mu_2,\ldots,\mu_\ell}(x) \to F^{A_1,A_2,\ldots,A_\ell}(P)$ where F obeys the following essential conditions:

1. $F(\lambda P) = \lambda^{-\Delta} F(P)$: F is degree $-\Delta$ in P,

2. Defined where $P^2 = 0$,

3. $P \cdot F = 0$: F is transverse to P_{A_i},

[9] This can be seen easily by computing $\frac{\partial^2 f(z)}{\partial z \cdot \partial z}$ and noting that the result will contain only contractions of the tensor f with all its different indices.

4. F is defined up to so-called "pure gauge" terms, which are terms proportional to P^{A_i}.

It is also implicit that F must inherit the same symmetries of f (that is, if f is symmetric and traceless, F will be as well). One can go back to physical space from embedding space by choosing the Poincaré section of the light cone $P^A = (1, x^2, x^\mu)$ and computing

$$f_{\mu_1,\ldots,\mu_\ell} = \frac{\partial P^{A_1}}{\partial x^{\mu_1}} \frac{\partial P^{A_2}}{\partial x^{\mu_2}} \cdots \frac{\partial P^{A_\ell}}{\partial x^{\mu_\ell}} F_{A_1,\ldots,A_\ell}. \tag{3.128}$$

Now, we perform the same trick as before in embedding space. We encode the (symmetric) tensor in terms of auxiliary vectors $Z^{A_1} \ldots Z^{A_\ell}$:

$$F(P;Z) \equiv F_{A_1,\ldots,A_\ell} Z^{A_1 \cdots} Z^{A_\ell}. \tag{3.129}$$

By analogy with the physical space picture, we may restrict traceless tensors to the region where $Z^2 = 0$. Moreover, since F is defined up to pure gauge terms, we are also afforded the liberty to drop terms that are proportional to $Z \cdot P$. A last consistency relation is to note that since F is of degree $-\Delta$ in P and $Z \cdot P = 0$, there exists a shift symmetry for $F(P;Z) \to F(P;Z+\lambda P)$. Adhering to the above relations and sanity checks, one can construct the basic building blocks of two and three point functions of fields with arbitrary spin.

Now, [2] further extends the above index-free formalism to the conformal partial waves/blocks. For external scalar operators, the basic idea behind a conformal block differs very little in four dimensions compared to the two dimensional case. One can still insert the identity operator and organize according to irreducible representations of the conformal group i.e. determine the projection of a conformal family (primary and its descendants) to the four point function, which defines the block. Another way is to apply the OPE algebra

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \sim \sum_{\Delta,\ell} \lambda_{\Delta,\ell} C(x-y, \partial_y)\mathcal{O}_2(y), \tag{3.130}$$

where $C(x-y, \partial_y)$ is determined completely by conformal invariance, twice to the four point function to obtain

$$\begin{aligned} \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle &= \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} \\ &\quad \times \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \sum_{\Delta,\ell} \lambda^{12}_{\Delta,\ell}\lambda^{34}_{\Delta,\ell} G_{\Delta,\ell}(u,v) \\ &\equiv \sum_{\Delta,\ell} \lambda^{12}_{\Delta,\ell}\lambda^{34}_{\Delta,\ell} W_{\Delta,\ell}(u,v), \end{aligned} \tag{3.131}$$

where u and v are the conformally invariant cross ratios (see Appendix F for a lengthier review), given by

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{3.132}$$

In four dimensions, the global conformal blocks $G_{\Delta,\ell}(u,v)$ (or equivalently, the partial waves $W_{\Delta,\ell}$) are given in terms of hypergeometric functions [86]. The idea of [2] is to use this result to determine the blocks associated with external operators with spin. In effect, the spin structure of a correlation function is propagated by certain derivative operators acting on the scalar blocks. Naturally, this will subject our correlation functions to the same caveats that grant a closed form expression for the scalar blocks (namely, that the exchanged operators are symmetric and traceless). Since the four point function of scalars is obtained by gluing together three point functions of spin $(0, 0, \ell)$, the task is to determine the “left” and ”right” differential operators that generate spinning operators J_i out of scalar operators in the three point functions

$$\langle J_1(P_1; Z_1) J_2(P_2; Z_2) \mathcal{O}(P; Z)\rangle = \mathcal{D}_{\text{left}} \langle \phi_1(P_1)\phi_2(P_2)\mathcal{O}(P; Z)\rangle \tag{3.133}$$

$$\langle J_3(P_3; Z_3) J_4(P_4; Z_4) \mathcal{O}(P; Z)\rangle = \mathcal{D}_{\text{right}} \langle \phi_3(P_3)\phi_4(P_4)\mathcal{O}(P; Z)\rangle, \tag{3.134}$$

such that the derivative operators propagate the index structure completely. The spin (ℓ_1, ℓ_2, ℓ_3) and $(0, 0, \ell)$ three point functions can be determined in full generality. Combined with certain consistency conditions, the derivative operators can also be determined without ambiguity. They are:

$$\begin{aligned} D_{11} \equiv &(P_1 \cdot P_2)(Z_1 \cdot \frac{\partial}{\partial P_2}) - (Z_1 \cdot P_2)(P_1 \cdot \frac{\partial}{\partial P_2}) \\ &- (Z_1 \cdot Z_2)(P_1 \cdot \frac{\partial}{\partial Z_2}) + (P_1 \cdot Z_2)(Z_1 \cdot \frac{\partial}{\partial Z_2}), \end{aligned} \tag{3.135}$$

$$D_{12} \equiv (P_1 \cdot P_2)(Z_1 \cdot \frac{\partial}{\partial P_1}) - (Z_1 \cdot P_2)(P_1 \cdot \frac{\partial}{\partial P_1}) + (Z_1 \cdot P_2)(Z_1 \cdot \frac{\partial}{\partial Z_1}), \tag{3.136}$$

$$D_{21} \equiv (P_2 \cdot P_1)(Z_2 \cdot \frac{\partial}{\partial P_2}) - (Z_2 \cdot P_1)(P_2 \cdot \frac{\partial}{\partial P_2}) + (Z_2 \cdot P_1)(Z_2 \cdot \frac{\partial}{\partial Z_2}), \tag{3.137}$$

$$\begin{aligned} D_{22} \equiv &(P_2 \cdot P_1)(Z_1 \cdot \frac{\partial}{\partial P_1}) - (Z_2 \cdot P_1)(P_2 \cdot \frac{\partial}{\partial P_1}) \\ &- (Z_2 \cdot Z_1)(P_2 \cdot \frac{\partial}{\partial Z_1}) + (P_2 \cdot Z_1)(Z_2 \cdot \frac{\partial}{\partial Z_1}). \end{aligned} \tag{3.138}$$

Along with one last trivial operator (in that does not affect the blocks) $H_{12} \equiv -2[(Z_1 \cdot Z_2)(P_1 \cdot P_2) - (Z_1 \cdot P_2)(Z_2 \cdot P_1)]$, these objects allow one to generate three point functions of spinning fields from a scalar-scalar-spin ℓ correlator. The notation denotes that the operator D_{ij} raises the spin at point i by one and lowers the scaling dimension at j by one.

3.6.2.2 Spin-1 Current

We will now apply the techniques we reviewed in §3.6.2.1 to the four point function consisting of a single spin-1 current and four other scalar operators, $\langle J_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$. In the differential basis of Eqs. (3.135) - (3.138), this correlation function is a linear combination of

$$D_{11}W_{\mathcal{O}}^{10}, \qquad D_{12}W_{\mathcal{O}}^{01}, \tag{3.139}$$

where $W_{\mathcal{O}}^{ij}$ is the usual scalar conformal partial wave with $\Delta_1 \to \Delta_1 + i$ and $\Delta_2 \to \Delta_2 + j$. Note that the derivatives with respect to Z_i vanish since the scalar partial waves only have dependence on the P_i. Therefore, acting on the scalar partial waves with the above derivative operators gives us

$$\begin{aligned} D_{11}W_{\mathcal{O}}^{10} =&(P_1\cdot P_2)\bigg[Z_1\cdot\frac{\partial\chi^{10}}{\partial P_2}G_{\mathcal{O}}^{10}+\chi^{10}\left(\frac{\partial G_{\mathcal{O}}^{10}}{\partial z}\lambda+\frac{\partial G_{\mathcal{O}}^{10}}{\partial\bar{z}}\bar{\lambda}\right)Z_1\cdot\frac{\partial v}{\partial P_2}\\ &+\chi^{10}\left(\frac{\partial G_{\mathcal{O}}^{10}}{\partial z}\mu+\frac{\partial G_{\mathcal{O}}^{10}}{\partial\bar{z}}\bar{\mu}\right)Z_1\cdot\frac{\partial u}{\partial P_2}\bigg]-(Z_1\cdot P_2)\left[Z_1\leftrightarrow P_1\right], \end{aligned}\tag{3.140}$$

$$\begin{aligned} D_{12}W_{\mathcal{O}}^{01} =&(P_1\cdot P_2)\bigg[Z_1\cdot\frac{\partial\chi^{01}}{\partial P_1}G_{\mathcal{O}}^{01}+\chi^{01}\left(\frac{\partial G_{\mathcal{O}}^{01}}{\partial z}\lambda+\frac{\partial G_{\mathcal{O}}^{01}}{\partial\bar{z}}\bar{\lambda}\right)Z_1\cdot\frac{\partial v}{\partial P_1}\\ &+\chi^{01}\left(\frac{\partial G_{\mathcal{O}}^{01}}{\partial z}\mu+\frac{\partial G_{\mathcal{O}}^{01}}{\partial\bar{z}}\bar{\mu}\right)Z_1\cdot\frac{\partial u}{\partial P_1}\bigg]-(Z_1\cdot P_2)\left[Z_1\leftrightarrow P_1\right], \end{aligned}\tag{3.141}$$

where χ is the pre-factor of the partial waves ($W_{\mathcal{O}}\equiv\chi G_{\mathcal{O}}$), and μ, $\bar{\mu}$, λ, and $\bar{\lambda}$ are functions of u and v defined in Appendix G. These functions arise because we have traded derivatives acting with respect to the P_i in favor of the variables z and $\bar{z}$, which are related to u and v via the relations:

$$u \equiv z\bar{z}, \tag{3.142}$$

$$v \equiv (1-z)(1-\bar{z}). \tag{3.143}$$

In parity with the two dimensional case, the ET will concern how derivatives act on the scalar blocks, and we will devote the remainder of this section to this, although we have computed the full result in Appendix I. For notational simplicity, we further introduce

$$\phi_Y^{ij,kl} \equiv Y_k \cdot \frac{\partial \chi^{ij}}{\partial P_l}, \qquad \zeta_{Y,x}^{mn} \equiv Y_m \cdot \frac{\partial x}{\partial P_n}. \tag{3.144}$$

With these definitions, let us focus on the action of D_{11} first. Denoting $\tilde{(\cdot)}$ as the projection to physical space (meaning that we project all the Z's and P's to z's and x_i's), we have

$$\begin{aligned} D_{11} W_{\mathcal{O}}^{10} = &-\frac{1}{2} x_{12}^2 \Bigg[\tilde{\phi}_Z^{10,12} G_{\mathcal{O}}^{10} + \chi^{10} \left(\frac{\partial G_{\mathcal{O}}^{10}}{\partial z} \tilde{\lambda} + \frac{\partial G_{\mathcal{O}}^{10}}{\partial \bar{z}} \tilde{\bar{\lambda}} \right) \tilde{\zeta}_{Z,v}^{12} \\ &+ \chi^{10} \left(\frac{\partial G_{\mathcal{O}}^{10}}{\partial z} \tilde{\mu} + \frac{\partial G_{\mathcal{O}}^{10}}{\partial \bar{z}} \tilde{\bar{\mu}} \right) \tilde{\zeta}_{Z,u}^{12} \Bigg] + z_1 \cdot x_{12} \left(\tilde{\phi}_Z^{10,12} \leftrightarrow \tilde{\phi}_P^{10,12}, \tilde{\zeta}_Z \leftrightarrow \tilde{\zeta}_P \right). \end{aligned} \tag{3.145}$$

We would like to calculate the divergence of the four point function. This is accomplished by first calculating $D_{11} W_{\mathcal{O}}^{10}$ and $D_{12} W_{\mathcal{O}}^{10}$, projecting to physical space, and then computing the action of the divergence $\mathcal{D}_c \equiv \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial z}$ (for an explanation of why this works, see Appendix H). When we act with $\mathcal{D}_c$, there will be *many* terms that are proliferated. It is therefore wise to systematically examine and categorize these terms in a qualitative way. It turns out that all terms can be classified as follows: (a) double and single derivative terms of the form $\partial^2 G_{\mathcal{O}}$ and $\partial G_{\mathcal{O}}$ (the derivatives act with respect to z or $\bar{z}$); (b) overall prefactors like $\tilde{\lambda}$ and $\tilde{\tilde{\lambda}}$ in Eq. (3.146) that are not relevant in determining if $\mathcal{D}_c\langle J\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ can be written in terms of scalar blocks; and (c) finite terms, which trivially become scalar functions multiplying scalar partial waves.

Two types of terms that will show up corresponding to category (a) are the double derivatives

$$\tilde{\lambda}\frac{\partial^2 G_{\mathcal{O}}}{\partial x_1 \partial z}, \qquad \tilde{\tilde{\lambda}}\frac{\partial^2 G_{\mathcal{O}}}{\partial x_1 \partial \bar{z}}. \tag{3.146}$$

As we alluded to earlier, the $d = 4$ the global conformal blocks are given in terms of hypergeometric functions,

$$G_{\mathcal{O}}(z,\bar{z}) = \frac{(-)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right], \tag{3.147}$$

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta; x\right), \tag{3.148}$$

with $\Delta_{ij} \equiv \Delta_i - \Delta_j$. The problem of trying to figure out how these derivatives act on the scalar blocks can be translated into figuring out how they act on k_β. Consider the derivative acting with respect to z first. Note that

$$\begin{aligned}\frac{\partial k_\beta}{\partial z} =& \frac{z^{\beta/2}}{4\beta}\Bigg[\frac{2\beta^2\,{}_2F_1\left(\frac{1}{2}(\beta-\Delta_{12}), \frac{1}{2}(\beta+\Delta_{34}); \beta; z\right)}{z} \\ &+ (\beta-\Delta_{12})(\beta+\Delta_{34})\,{}_2F_1\left(\frac{1}{2}(\beta-\Delta_{12}+2), \frac{1}{2}(\beta+\Delta_{34}+2); \beta+1; z\right)\Bigg] \\ =& \frac{\beta}{2z}k_\beta(z) + \frac{(\beta-\Delta_{12})(\beta+\Delta_{34})}{4\beta\sqrt{z}}k_{\beta+1}^{\Delta_1\to\Delta_1-1,\Delta_3\to\Delta_3+1}(z),\end{aligned} \tag{3.149}$$

and so we obtain what looks almost like a recursion relation

$$\begin{aligned}\frac{\partial G_{\Delta,l}}{\partial z} =& -\frac{\bar{z}}{z(z-\bar{z})}G_{\Delta,l} + \frac{(-)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}} \\ &\times \left[\frac{(\Delta+l)k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (\Delta-l-2)k_{\Delta+l}(\bar{z})k_{\Delta-l-2}(z)}{2z}\right. \\ &+ M_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z})k^{\Delta_1\to\Delta_1-1,\Delta_3\to\Delta_3+1}_{\Delta+l+1}(z) \\ &\left.- M_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})k^{\Delta_1\to\Delta_1-1,\Delta_3\to\Delta_3+1}_{\Delta-l-1}(z)\right], \end{aligned} \tag{3.150}$$

where we have defined $M_\beta(z) \equiv (\beta-\Delta_{12})(\beta+\Delta_{34})/(4\beta\sqrt{z})$. The above result is a function of z and $\bar{z}$ so we should write

$$\frac{\partial^2 G_{\mathcal{O}}}{\partial x_1 \partial z} = \frac{\partial^2 G_{\mathcal{O}}}{\partial z^2}\left(\tilde{\lambda}\frac{\partial v}{\partial x_1} + \tilde{\mu}\frac{\partial u}{\partial x_1}\right) + \frac{\partial^2 G_{\mathcal{O}}}{\partial \bar{z}\partial z}\left(\tilde{\tilde{\lambda}}\frac{\partial v}{\partial x_1} + \tilde{\tilde{\mu}}\frac{\partial u}{\partial x_1}\right), \tag{3.151}$$

so we have to consider the action of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ on Eq. (3.150). Let's first look at what happens when we apply $\frac{\partial}{\partial z}$ to $\frac{\partial G}{\partial z}$:

$$\begin{aligned}\frac{\partial^2 G_{\Delta,l}}{\partial z^2} =& \frac{\bar{z}}{4z\,(z-\bar{z})^2}\Bigg\{8G_{\Delta,l}(z,\bar{z}) \\ &+ \frac{(-)^l}{2^l}\Bigg(k_a(\bar{z})\Big[4z^2\,(\bar{z}-z)\,M_b(z)M_{b+1}(z)k^{\Delta_1',\Delta_3'}_{b+2}(z) - 2z\,(z-\bar{z})\,(b+1)k_{b+1}(z)M_b(z)+ \\ &- k^{\Delta_1',\Delta_3'}_{b+1}(z)M_b(z)\left[z(b-1)+\bar{z}(b+3)\right] + b\left[\bar{z}(b+2)-z(b-2)\right]k_b(z)\Big] \\ &+ k_b(\bar{z})\Big[-4z^2\,(\bar{z}-z)\,M_a(z)M_{a+1}(z)k^{\Delta_1',\Delta_3'}_{a+2}(z) + 2z\,(z-\bar{z})\,(a+1)k_{a+1}(z)M_a(z) \\ &+ k^{\Delta_1',\Delta_3'}_{a+1}(z)M_a(z)\left[z(a-1)+\bar{z}(a+3)\right] - a\left(\bar{z}(a+2)-z(a-2)\right)k_a(z)\Big]\Bigg)\Bigg\}, \end{aligned} \tag{3.152}$$

where we have defined $a \equiv \Delta + l$ and $b \equiv \Delta - l - 2$.

The above result is one of three types of second-derivative terms that appears in $\mathcal{D}_c\langle J\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ (the other two should be the second derivatives in $\bar{z}$ and the mixed derivatives in z and $\bar{z}$). Now, no approximations have been used so far, so the result we just obtained looks rather abstruse. Evidently, there are explicit scalar blocks and there are other terms that do not resemble the scalar blocks at first glance. Again, in analogy with the two dimensional case, we can look at the large Δ limit (that is $\Delta \gg \Delta_1 \equiv \Delta_J$) of each term. This limit is interesting because we know that the twists $\tau = \Delta - l$ play the role of center-of-mass energies in the conformal partial waves. Therefore, when the center of mass energy is large for the 2 to 2 scattering in AdS, we expect the equivalence theorem to hold. Before we apply this limit to the above result, we first see that all the terms come in the with "right" sign and can be grouped in the form

$$k_a(z)k_b(\bar{z}) - k_a(\bar{z})k_b(z). \tag{3.153}$$

However, there are two problems to deal with first: (1) we see that taking the large Δ limit is not enough to reproduce the scalar blocks (because a and b are different coefficients) and (2) there are shifted scalar blocks (shifted in the sense of $\Delta_1 \to \Delta_1 - 1$). The first problem can easily be dealt with by making a further assumption: $\Delta \gg l, 1$ such that $a = \Delta + l \approx b = \Delta - l - 2$. This simply translates to the fact that exchanged bulk fields do not carry large angular momentum, which would imply a large impact parameter and lower the overall energy carried by the gauge boson. For the ET theorem to hold, the energy carried by the bulk gauge boson must be large and the CFT reflects this property.

The second condition comes from the presence of shifted k-functions, when we interpreted $\frac{\partial k_\beta}{\partial z}$ as a recursion relation which relates the shifted $\beta \to \beta+1$ k-functions to k_β. The only way to do this was to assume that there was a unit dimension shift to either Δ_1 *and* Δ_3 *or* a unit dimension shift to Δ_2 *and* Δ_4. However, because we choose $\Delta \gg 1$, these small shifts to the external operator dimensions can safely be absorbed into Δ. The shifted terms then also combine in a simple way to produce scalar blocks in the large Δ limit. Another interpretation of this condition is to look at higher spin correlators. There, it turns out that the condition is really for $\Delta_{1,3} \gg s_{1,3}$ i.e. we do not consider an outlandish scenario where the twists of the external operators are dominated by their spins (this is simply because each derivative acting on a scalar block essentially becomes a factor of Δ in the large Δ limit and higher spin currents in correlation functions are written in the differential basis as more derivative operators acting on scalar partial waves).

Therefore, given the existence of the OPE where exchanged operators have twist $\tau = \Delta - \ell$ and are symmetric traceless, and the usual reasonable assumptions about a CFT$_4$, the following conditions are necessary and sufficient in order for $\mathcal{D}_c\langle J_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$ (the scalar operators $\mathcal{O}_i$ may all be different) to admit scalar modes consistent with the Goldstone equivalence theorem:

1. $\Delta \gg \Delta_{J_1}$,

2. $\Delta \gg l$, 1.

One might be concerned that so far that our analysis has only focused on the derivative terms that act with respect to z. There are certainly many more terms that appear in the divergence, but these terms do not refute the above result; the above conditions are necessary and sufficient. For example we have the following result for the $\bar{z}$ term:

$$\begin{aligned}\frac{\partial^2 G}{\partial \bar{z}^2} = \frac{1}{4\left(z-\bar{z}\right)^2 \bar{z}} \Bigg\{ & 8zG_{\Delta,l}(z,\bar{z}) \\ & + \frac{(-)^l}{2^l}\Bigg[\bar{z}k_b(z)\Bigg(a^2\left(\bar{z}-z\right)k_a(\bar{z}) + 2\bar{z}\Big(\left(\bar{z}-z\right)M_a(\bar{z})\big((a+1)k_{a+1}(\bar{z}) \\ & + 2k_{a+2}^{\Delta_1',\Delta_3'}(\bar{z})M_{a+1}(\bar{z})\big) + k_{a+1}^{\Delta_1',\Delta_3'}(\bar{z})M_a(\bar{z})\left[\bar{z}(a+1)-z(a+5)\right]\Big)\Bigg) \\ & + \bar{z}k_a(z)\Bigg(-b^2\left(\bar{z}-z\right)k_b(\bar{z}) - 2\bar{z}\Big(\left(\bar{z}-z\right)M_b(\bar{z})\big((b+1)k_{b+1}(\bar{z}) \\ & + 2k_{b+2}^{\Delta_1',\Delta_3'}(\bar{z})M_{b+1}(\bar{z})\big) + k_{b+1}^{\Delta_1',\Delta_3'}(\bar{z})M_b(\bar{z})\left[\bar{z}(b+1)-z(b+5)\right]\Big)\Bigg) \\ & + 4z\bar{z}\left(bk_b(\bar{z})k_a(z) - ak_b(z)k_a(\bar{z})\right)\Bigg]\Bigg\}. \end{aligned} \tag{3.154}$$

The above result becomes a scalar block in the limits previously considered. There is also the mixed derivative term $\partial_z\partial_{\bar{z}}G_{\Delta,l}$ and it is straightforward to verify that it, too, adheres to this behavior (although of course there the result will contain a many more terms). We compute the full result of $\mathcal{D}_c\langle J_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}(x_4)\rangle$ in Appendix I and verify that in the limit $\Delta \gg \Delta_J, l$ that the correlator reduces to the form

$$\begin{aligned}\mathcal{D}_c\langle J_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}(x_4)\rangle \approx &(f_1+f_1')\sum_{\Delta,l}\lambda^{12}_{\Delta,l}\lambda^{34}_{\Delta,l}F_{1,\Delta}G_{\Delta,l}\\ &+(f_2+f_2')\sum_{\Delta,l}\lambda^{12}_{\Delta,l}\lambda^{34}_{\Delta,l}F_{2,\Delta}G_{\Delta,l}\\ &+(f_3+f_3')\sum_{\Delta,l}\lambda^{12}_{\Delta,l}\lambda^{34}_{\Delta,l}F_{3,\Delta}G_{\Delta,l}\\ &+\text{ terms of order }\Delta\text{ and below,}\end{aligned} \tag{3.155}$$

where F_i are proportional to Δ^2 and also depend on the coordinates of the external operators through z and $\bar{z}$. The idea then is that when one rescales the partial wave coefficients uniformly by $\lambda^{ij}_{\Delta,l}\to\Delta^{-1}\lambda^{ij}_{\Delta,l}$, then it is clear that the leading order terms will be *scalar correlation* functions, modulo the prefactor functions. This is a subtle consequence of the conformal *blocks* satisfying the ET.

To summarize, we found that

- the divergence of the four point function could be written in the differential basis as Eqs. (3.135) - (3.138).
- After some massaging of terms, the divergence could be expressed as

$$\begin{aligned}\partial\cdot G_1 =&f_1\partial^2G^{10}+f_1'\partial^2G^{01}+f_2\partial\bar{\partial}G^{10}+f_2'\partial\bar{\partial}G^{01}\\ &+f_3\bar{\partial}^2G^{10}+f_3'\bar{\partial}^2G^{01}+g_1\partial G^{10}+g_1'\partial G^{01}\\ &+g_2\bar{\partial}G^{10}+g_2'\bar{\partial}G^{01}+hG^{10}+h'G^{01},\end{aligned} \tag{3.156}$$

where the derivatives are either with respect to z (∂) or $\bar{z}$ ($\bar{\partial}$). The functions f_i, g_i, h depend only on the dimensions and coordinates of the external operators.

- In the large Δ limit, taking both $\Delta \gg l$, Δ_J, we found that it reduces further to Eq. (3.155). The functions $F_{i,\Delta}$ are all $\sim \Delta^2$ to leading order and are also given in the Appendix.

- This implies that the blocks for the four point function become scalar blocks, assuming that the exchanged operators are symmetric and traceless.

3.7 Discussion

We have examined the ET as a statement about propagating AdS massive gauge bosons and the corresponding CFT currents.

In the AdS bulk, we have defined an analogue to the S-matrix as a correlation function of creation and annihilation operators, and showed a relationship between such objects that involve z "polarized" gauge bosons (in the Poincaré patch) and their corresponding Goldstone boson when the magnitude of its momentum is large. It was shown that these matrix elements naturally satisfy the AdS ET regardless of the external momentum scales when the conformal dimensions of the leading order exchanged particles are sufficiently larger than the dimension of the gauge boson. This follows since arbitrarily increasing the exchanged scaling dimensions arbitrarily suppresses the interacting piece of the matrix element, which is compensated for by increasing the incoming momentum to the point that the equivalence theorem is satisfied anyway. As a consequence, the divergence of conformal currents dual to the gauge boson in correlators for theories with "heavy" exchange operators is approximately primary. Indeed, when the correlators are expressed as integrals of matrix elements, only the upper part of integration space contributes for the interacting pieces.

On the side of the CFT, we have shown that the conformal blocks for a correlator of a spin-1 current and three scalars satisfy the ET in the large twist limit. It would be a natural extension of this work to generalize the result to non symmetric tensors, making use of the "shadow" formalism of [90], which offers an alternative method to treat spinning fields in full generality.

It is also worthwhile to note that there are other interpretations of center of mass energies in a CFT. In flat space, the equivalence theorem is a statement of scattering amplitudes in a large momentum limit. For CFT correlation functions, the analog of momentum space is Mellin space [91–94]. It would be interesting to see if the ET can be obtained as a kinematic limit in Mellin space, following the derivation in flat space.

In our treatment of CFT currents and conformal blocks, we did not mention higher spin broken currents in four dimensions. The reason why such an analysis is difficult is because the number of derivative terms needed to classify higher spin correlation functions grows with the spin of the current. Therefore, computing higher spin correlation functions in terms of their conformal blocks becomes an exercise in computing many derivative expansions. In principle, one could use the index-free approach to understand the structure of higher spin broken currents [95, 96]. It would also be interesting, but difficult, to see show explicitly the presence of lower-spin terms at sub-leading order in Δ/Δ_J in the conformal blocks for higher spin correlation functions.

Appendix A

Classical fields with multi-trace deformations

We can extend the formalism of §2.3 to determine the bulk-boundary propagator with multi-trace boundary deformations. Consider the following boundary deformation and associated boundary conditions:

$$\mathcal{L}_b = \epsilon^{d-\Delta+1}\phi_b\phi + \frac{1}{n}\lambda\epsilon^{d-n\Delta+1}\phi^n \implies \delta B[\phi] = -\epsilon^{d-\Delta}\phi_b - \lambda\epsilon^{d-\Delta}\alpha^{n-1}. \tag{A.1}$$

The usual procedure results in

$$b(x) = \frac{1}{2\nu}\left[\phi_b(x) + \lambda\alpha^{n-1}(x)\right], \tag{A.2}$$

and we face solving

$$\alpha(x) = \int d^dx' \int đ^dp \, \frac{\Gamma(\nu)}{\Gamma(1-\nu)}\left(\frac{p}{2}\right)^{-2\nu}\frac{1}{2}\left[\phi_b(x') + \lambda\alpha^{n-1}(x')\right]e^{ip\cdot(x-x')} \tag{A.3}$$

for α in terms of ϕ_b. Determining α exactly seems like a hopeless endeavor, but we can still compute the contribution of the deformation to the bulk-boundary propagator. Given the nonlinearity we should expect these deformations to generate interaction terms in higher-point correlation functions. Classically, the interacting piece of N-point functions can be computed as contributions to the Nth harmonic of the source. This follows explicitly from

$$\langle \alpha(x_1)\ldots\alpha(x_N)\rangle \propto \frac{\delta}{\delta\phi_b(x_1)}\cdots\frac{\delta}{\delta\phi_b(x_N)} e^{\int d^d y\,\alpha(y)\phi_b(y)}\bigg|_{\phi_b=0}, \tag{A.4}$$

the purely interacting piece of which is

$$\frac{\delta}{\delta\phi_b(x_2)}\cdots\frac{\delta}{\delta\phi_b(x_N)}\alpha(x_1)+\cdots+\frac{\delta}{\delta\phi_b(x_1)}\cdots\frac{\delta}{\delta\phi_b(x_{N-1})}\alpha(x_N)\bigg|_{\phi_b=0}. \tag{A.5}$$

We can build an expansion of Eq.(A.3) in nested functionals of ϕ_b, which poises us perfectly to compute N-point functions as sums of products of integral kernels in accordance with Eqs.(A.4)&(A.5). The bulk-boundary propagator would then simply be the first functional derivative of ϕ with respect to ϕ_b. Since $\alpha[\phi_b = 0] = 0$, there are no classical contributions to the bulk-boundary propagator for $n > 2$.

Computing higher point correlation functions in AdS generated by boundary deformations, of course, require us to consider bulk sources. We can expect to use similar methods to determine these. Heuristically, even classically an N-point function in the bulk should involve an N-point function at the boundary, and, from Eq.(A.3), we expect a non-vanishing tree-level correlator only for $N = n$. There is more in the details, but at the level of the bulk-boundary propagator, we can stop here.

In the absence of boundary deformations, the wave functions are well known [97,98]:

$$f^0(\vec{p}, m; x, z) = \sqrt{\frac{im}{2p_{m0}}} z^{\frac{d}{2}} J_{-\nu}(mz) e^{ip_m \cdot x}, \tag{A.6}$$

where $p_{m0} = i\sqrt{\vec{p}^2 + m^2}$. As was the case with the bulk-boundary propagator, we can use this knowledge to compute the modified wave functions in the presence of boundary deformations.

The procedure works as before with $\phi \to f$, except we exclude boundary source terms from our Lagrangian and boundary conditions,

$$\mathcal{L}_b = \frac{1}{n}\lambda\epsilon^{d-n\Delta+1}\phi^n \implies \delta B[\phi] = -\lambda\epsilon^{d-\Delta}\alpha^{n-1}, \tag{A.7}$$

and we seek homogeneous solutions. Now, Eq.(A.3) becomes

$$\begin{aligned} \alpha(x) =& \sqrt{\frac{im}{2p_{m0}}} \frac{1}{\Gamma(1-\nu)} \left(\frac{m}{2}\right)^{-\nu} e^{ip_m \cdot x} \\ &+ \int d^d x' \int d\!\!\!{}^-{}^d p \, \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \left(\frac{p}{2}\right)^{-2\nu} \frac{\lambda}{2} \alpha^{n-1}(x') e^{ip\cdot(x-x')}. \end{aligned} \tag{A.8}$$

For $n = 2$, this is straight forward:

$$\alpha(x) = \frac{1}{1 - \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \left(\frac{p_m}{2}\right)^{-2\nu} \frac{\lambda}{2}} \sqrt{\frac{im}{2p_{m0}}} \frac{1}{\Gamma(1-\nu)} \left(\frac{m}{2}\right)^{-\nu} e^{ip_m \cdot x}. \tag{A.9}$$

And so

$$\begin{aligned} f(x, z) =& \sqrt{\frac{im}{2p_{m0}}} z^{\frac{d}{2}} \Bigg[J_{-\nu}(mz) \\ &+ \frac{\lambda}{1 - \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \left(\frac{p_m}{2}\right)^{-2\nu} \frac{\lambda}{2}} \frac{1}{\Gamma(1-\nu)^2} \left(\frac{m}{2}\right)^{-\nu} \left(\frac{p_m}{2}\right)^{-\nu} K_\nu(p_m z) \Bigg] e^{ip_m \cdot x}. \end{aligned} \tag{A.10}$$

It is worthwhile to point out that Eq.(A.8) requires α to be classical at the boundary, demonstrating that the classical methods sufficient for double-trace deformations are inapplicable for more general deformations.

Appendix B

Spontaneously Broken Gauge Lagrangian

Here, we justify the form of the Lagrangian in Eq. (3.14). In this section, overbars indicate a vector under a gauge group; the same symbol without an overbar is the magnitude of the corresponding vector. The '$\cdot$' symbol indicates a sum over all gauge indices; the '$\times$' symbol indicates a sum over only broken gauge indices; the '$*$' symbol indicates a sum over all gauge indices excluding the broken ones.

Consider an $SU(N)$ gauge theory coupled to a charged scalar, $\bar{\Phi}$, that acquires a vacuum expectation value, $\frac{\bar{v}}{\sqrt{2}}$:

$$\mathcal{L} \supset |(\partial - igA \cdot T)\bar{\Phi}|^2 - V(|\Phi|^2). \tag{B.1}$$

The Higgs mechanism breaks an $SU(N)$ symmetry down to an $SU(N-1)$ symmetry, resulting in $2N-1$ Goldstone bosons. Since Φ originally had $2N$ degrees of freedom, only one Higgs-like degree of freedom remains. Φ can thus be parameterized as

$$\bar{\Phi} = \exp\left[i\frac{\pi}{v} \times T\right](\bar{v}/\sqrt{2} + \bar{\phi}), \tag{B.2}$$

where the π's are the Goldstone bosons. Neither $\bar{v}$ nor $\bar{\phi}$ can be annihilated by the generators in the exponential, so we conclude $\bar{\phi} \propto \bar{v}$.

The kinetic term in Eq. (B.1) can then be written as

$$\begin{aligned}
\left|(\partial - igA\cdot T)e^{\left[i\frac{\pi}{v}\times T\right]}(\bar{v}+\bar{\phi})\right|^2 =& \left|e^{\left[i\frac{\pi}{v}\times T\right]}\partial\bar{\phi} - ig(A - m_A^{-1}\partial\pi)\times Te^{\left[i\frac{\pi}{v}\times T\right]}(\bar{v}/\sqrt{2}+\bar{\phi})\right. \\
&\left. - igA * Te^{\left[i\frac{\pi}{v}\times T\right]}(\bar{v}/\sqrt{2}+\bar{\phi})\right|^2 & \text{(B.3)} \\
=& \left|\partial\bar{\phi} - ig(A - m_A^{-1}D\pi)\times T(\bar{v}/\sqrt{2}+\bar{\phi})\right|^2 & \text{(B.4)} \\
=& \frac{1}{2}m_A^2(A_{\text{broken}} - m_A^{-1}D\pi)^2 \\
&+ \mathcal{L}_{int}(\frac{1}{2}(A_{\text{broken}} - m_A^{-1}D\pi)^2, \phi), & \text{(B.5)}
\end{aligned}$$

where $m_A \equiv gv$. The second line follows from the first since two generators from the broken part of $SU(N)$ commute into the preserved $SU(N-1)$ part, $[A\times T, e^{\left[i\frac{\pi}{v}\times T\right]}]^c = -m_A^{-1}ge^{\left[i\frac{\pi}{v}\times T\right]}A^a\pi^b f^{abc}T^c$ for structure constants f^{abc}, and $A * T\bar{v} = 0$; the third line foliows since $T\times T = \mathbb{I}$. The important upshot is that the Lagrangian for an $SU(N)$ gauge theory broken by the Higgs mechanism satisfies the form given in Eq. (3.14). This form holds for any spontaneous breaking mechanism and follows generally by simply demanding the Goldstone bosons transform as simple shifts, are derivatively coupled, and that gauge symmetry should still hold at the Lagrangian level.

Appendix C

Review of the Schwinger-Dyson Equations

The following is an adaptation of the approach [99] takes to derive the Schwinger-Dyson equations. Consider a general path integral for some fields $\{\phi_a\}$ of arbitrary spin in $d+1$ dimensions,

$$Z[\phi_a, J] = \int \mathcal{D}\phi_a e^{-i\left(S[\phi_a] - \int d^{d+1}x\, J_a \cdot \phi_a\right)}, \tag{C.1}$$

where '$\cdot$' indicates a contraction of all indices between J and ϕ. Now vary the fields in a manner commensurate with the path integral measure, $\phi_a \to \phi_a + \delta\phi_a$, and consider the resulting path integral,

$$Z[\phi_a + \delta\phi_a, J] = \int \mathcal{D}\phi_a e^{-i\left(S[\phi_a] - \int d^{d+1}x\, J_a\cdot\phi_a\right)} e^{-i\left(\int d^{d+1}x \frac{\delta S[\phi_a]}{\delta\phi_a}\cdot\delta\phi_a - \int d^{d+1}x\, J_a\cdot\delta\phi_a\right)} \quad \text{(C.2)}$$

$$\approx \int \mathcal{D}\phi_a e^{-i\left(S[\phi_a] - \int d^{d+1}x\, J_a\cdot\phi_a\right)} \left[1 - i\int d^{d+1}x \left(\frac{\delta S[\phi_a]}{\delta\phi_a} - J_a\right)\cdot\delta\phi_a\right]. \quad \text{(C.3)}$$

Any transformation of fields that leaves the path integral measure invariant should leave the path integral itself invariant since fields are being integrated over all possible values anyway. Since this was exactly our constraint on the transformation of the fields, we see

$$Z[\phi_a + \delta\phi_a, J] = Z[\phi_a, J] \quad \text{(C.4)}$$

$$\Longrightarrow \int \mathcal{D}\phi_a e^{-i\left(S[\phi_a] - \int d^{d+1}x\, J_a\cdot\phi_a\right)} \left[\frac{\delta S[\phi_a]}{\delta\phi_a} - J_a\right] = 0. \quad \text{(C.5)}$$

Acting on both sides of Eq. (C.5) with $\prod_i \left(-i\frac{\delta}{\delta J_{a_i}(x_i)}\right)$ then setting $J_a = 0$ yields

$$\left[\int \mathcal{D}\phi_a e^{-iS[\phi_a]}\right]^{-1} \int \mathcal{D}\phi_a e^{-iS[\phi_a]} \frac{\delta S[\phi_a]}{\delta\phi_a(x)} \prod_i \phi_{a_i}(x_i) = \langle T \frac{\delta S[\phi_a]}{\delta\phi_a(x)} \prod_i \phi_{a_i}(x_i)\rangle$$

$$= -i\sum_i \delta^{d+1}(x - x_i). \quad \text{(C.6)}$$

By applying the differential operator associated with the classical equations of motion of a field to each field in a correlation function, the Schwinger-Dyson equations in Eq. (C.6) provide a tower of coupled differential equations that describe time ordered correlation functions sourced by contact terms (the delta functions).

Now consider the Lagrangian provided in Eq. (3.17). The resulting Schwinger-

Dyson equations for the broken gauge bosons and associated Goldstones are

$$[\nabla_N \nabla^M - \xi^{-1} \nabla^M \nabla_N - ({\nabla^2}^M_N + m_A^2 \delta^M_N)] \langle T A^{aN} \ldots \rangle = \langle T(A^{aM} - m_5^{-1} \partial^M \pi^a) \mathcal{L}'_{int} \ldots \rangle + \langle J^M \rangle + C_G \tag{C.7}$$

$$(\nabla^2 + \xi m_A^2) \langle T \pi^a \ldots \rangle = -m_A^{-1} \nabla_M \langle T(A^{aM} - m_5^{-1} \partial^M \pi^a) \mathcal{L}'_{int} \ldots \rangle + C_{GS} \tag{C.8}$$

where the C's are contact terms, $\mathcal{L}'_{int}$ is the derivative of $\mathcal{L}_{int}$ with respect to its first argument, and J^M is a conserved current to which the gauge fields couple, $J^{aM} = \frac{\partial}{\partial {A^a}_M} [\mathcal{L}_{G,int}[A^a] + \mathcal{L}_{GH}]$. The '$\ldots$' include other field operators.

Acting on both sides of Eq. (C.7) with $m_A^{-1} \nabla_M$ annihilates the conserved current and yields Eq. (3.20):

$$m_A^{-1} \nabla_M [\nabla_N \nabla^M - \xi^{-1} \nabla^M \nabla_N - ({\nabla^2}^M_N + m_A^2 \delta^M_N)] \langle T A^{aN} \ldots \rangle = -[\nabla^2 + \xi m_5^2] \langle T \pi^a \ldots \rangle. \tag{C.9}$$

Appendix D

Review of AdS Wave Functions

It is well known that a (real) scalar field can be expanded in terms of eigenfunctions that satisfy the wave equation for the spacetime in which it dwells as

$$\phi(x) = \sum_i \left[a_i f_i^\dagger(x) + h.c. \right], \tag{D.1}$$

where

$$\left[\nabla^2 + m_\phi^2 \right] f_i(x) = 0. \tag{D.2}$$

Here, i is simply used as a schematic label for the eigenfunctions and can generally be discrete or continuous and represent many parameters. In Poincaré patch coordinates in AdS the expansion becomes what is seen in Eq. (3.32) with wave functions given by

$$f(\vec{p}, m) = N_\phi z^{\frac{d}{2}} J_{\Delta_\phi - \frac{d}{2}}(mz) e^{-ip_m \cdot x} \tag{D.3}$$

where N_ϕ is a normalization factor. Defining the inner product on function space over constant-time slices of AdS as

$$\langle \Psi_1, \Psi_2 \rangle = i \int d^d x \, \sqrt{g} \, g^{00} \left[\Psi_1^\dagger \partial_0 \Psi_2 - \partial_0 \Psi_2^\dagger \Psi_1 \right] \tag{D.4}$$

and demanding $\langle f_m, f_n \rangle = \delta_{mn}$ yields the normalization $N_\phi = \frac{\sqrt{m}}{\sqrt{2p_{m0}}}$, resulting in wave functions given by Eq. (3.33).

Gauge fields can be expanded in a similar manner, but solving for their corresponding wave functions is slightly more involved. Since $A \in \mathcal{H} \otimes \mathcal{H}^* \otimes \mathbb{R}^{d+1} \otimes [C^2]^{d+1}$ is Hermitian, we can always expand the gauge field in terms of eigenfunctions as:

$$A^a_M = \sum_{s,i} \left[a^a_{s,i} h^\dagger_{s,iM}(x) + h.c. \right], \tag{D.5}$$

where $a_{s,i} \in \mathcal{H} \otimes \mathcal{H}^*$ and $h_{s,iM}(x) \in \mathbb{C}^{d+1} \otimes [C^2]^{d+1}$. The index '$s$' labels our basis in $\mathbb{C}^{d+1}$, thereby taking a value in a discrete, finite set. Demanding that the fields satisfy the Heisenberg equations of motion, we must choose $h_{s,iM}$ that have the same Casimir weight as the free fields. They must thus satisfy the free classical equations of motion for a massive gauge field:

$$\left[\nabla_N \nabla^M - \xi^{-1} \nabla^M \nabla_N - \left(\nabla^2 + m_A^2 \right) \delta^M_N \right] h^N_{s,i}(x) = 0. \tag{D.6}$$

Taking the covariant divergence of Eq. (D.6) and defining
$\tilde{h}_{s,i}(x) \equiv (\xi m_A)^{-1} \nabla_M h^M_{s,i}(x)$ yields

$$\left[\nabla^2 + \xi m_A^2 \right] \tilde{h}_{s,i}(x) = 0, \tag{D.7}$$

so the divergence of the vector wave function obeys the scalar wave equation. The vector wave function with the divergence degree of freedom projected out, $\bar{h}_{s,iM}(x) \equiv \left(\delta_M^N - \partial_M \nabla^{-2} \nabla^N\right) h_{s,iM}(x)$, then satisfies

$$\left[\nabla_N \nabla^M - \left(\nabla^{2}{}_N^M + m_A^2 \delta_N^M\right)\right] \bar{h}_{s,i}^N(x) = 0. \tag{D.8}$$

The full wave function is then constructed by solving Eqs. (D.7) and (D.8) and writing $h_{s,i}^M(x) = \partial^M \nabla^{-2} \tilde{h}_{s,i}(x) + \bar{h}_{s,i}^M(x)$.

Since the set $\{h_s\}$ is linearly independent and each h_s can be decomposed into a linear combination of divergenceless degrees of freedom and a scalar divergence, we can select the set such that $s = \xi$ contains only the scalar degree of freedom and all others are divergenceless. Additionally, we can select the set such that among the divergenceless wave functions only h_z has a nonvanishing z-component. Under this prescription, the operator in Eq. (D.8) can be diagonalized, leading to the expansion of the gauge fields to become what is seen in Eq. (3.31) with wave functions given by Eqs. (3.34) and (3.35):

$$h_{s,M}(\vec{p}, m) \underset{s \neq z,\xi}{=} \begin{cases} 0, & M = z \\ N_A \epsilon_{s,\mu}(\vec{p}) z^{\frac{d}{2}-1} J_{\Delta - \frac{d}{2}}(mz) e^{-ip_m \cdot x}, & M = \mu \end{cases}, \tag{D.9}$$

$$h_{z,M}(\vec{p}, m) =$$
$$\begin{cases} N_A z^{\frac{d}{2}} J_{\Delta - \frac{d}{2}}(mz) e^{-ip_m \cdot x}, & M = z \\ -i N_A \frac{p_{m\mu}}{m^2} \left[m z^{\frac{d}{2}} J_{\Delta - \frac{d}{2}+1}(mz) - [\Delta - (d-1)] z^{\frac{d}{2}-1} J_{\Delta - \frac{d}{2}}(mz) \right] e^{-ip_m \cdot x}, & M = \mu \end{cases}. \tag{D.10}$$

Defining the inner product on vector function space as

$$\langle \Psi_1, \Psi_2 \rangle = i \int d^d x \, \sqrt{g} \, g^{00} \Big\{ g^{MN} \Big[\Big(\Psi_{1M}^\dagger \nabla_0 \Psi_{2N} - \nabla_0 \Psi_{1M}^\dagger \Psi_{2N} \Big) - \Big(\Psi_{1M}^\dagger \nabla_N \Psi_{20} - \nabla_M \Psi_{10}^\dagger \Psi_{2N} \Big) \Big] + \xi^{-1} \Big(\Psi_{10}^\dagger \nabla^M \Psi_{2M} - \nabla^M \Psi_{1M}^\dagger \Psi_{20} \Big) \Big\} \tag{D.11}$$

and demanding $\langle h_{s,i}, h_{s',j} \rangle = \delta_{ss'} \delta_{ij}$ yields the normalization $N_A = \frac{\sqrt{m}}{\sqrt{2 p_{m0}}}$, resulting in vector wave functions given by Eqs. (3.34) and (3.35).

Appendix E

Review of the LSZ Formula

Obtaining the ET in AdS required us to consider matrix elements and exploit the upshot of the Schwinger-Dyson equations within an LSZ integral. To determine the LSZ-like integral on a curved spacetime, consider the following simple matrix element,

$$\langle T a_{s,\vec{p},m}(+\infty) a^{\dagger}_{s,\vec{p},m}(-\infty)\rangle. \tag{E.1}$$

To find a functional form of Eq. (E.1), we write

$$\begin{aligned}
&a_{s,\vec{p},m}(+\infty) = a_{s,\vec{p},m}(-\infty) + [\langle h_{s,\vec{p},m}(+\infty), A(+\infty)\rangle - \langle h_{s,\vec{p},m}(-\infty), A(-\infty)\rangle] \\
&= a_{s,\vec{p},m}(-\infty) \\
&+ i\int_{-\infty}^{+\infty} dt \int d^d x\, \sqrt{g}\, h^{\dagger}_{s,\vec{p},m,M} \left[\left({\nabla^2}^M_N + m_A^2 \delta^M_N\right) - \nabla_N \nabla^M + \xi^{-1}\nabla^M\nabla_N\right] A^N,
\end{aligned} \tag{E.2}$$

where the integral expression in the second line follows from the first by using the definition of the vector function space inner product in Eq. (D.11) and noting that the differential operator appearing in the second line annihilates $h_{s,\vec{p},m}$.

We find a similar expression for $a^{\dagger}_{s,\vec{p},m}(-\infty)$:

$$\begin{aligned} a^{\dagger}_{s,\vec{p},m}(-\infty) &= a^{\dagger}_{s,\vec{p},m}(+\infty) \\ &- i\int_{-\infty}^{+\infty} dt \int d^d x\, \sqrt{g}\, h_{s,\vec{p},mM} \left[\left(\nabla^{2}{}^{M}_{N} + m_A^2 \delta^M_N\right) - \nabla_N \nabla^M + \xi^{-1}\nabla^M \nabla_N\right] A^N. \end{aligned} \tag{E.3}$$

Inserting Eqs. (E.2) and (E.3) back into Eq. (E.1) yields

$$\begin{aligned} \langle T a_{s,\vec{p},m}(+\infty) a^{\dagger}_{s,\vec{p},m}(-\infty)\rangle = (i)^2 \int dx\, dx'\, \sqrt{g(x)g(x')}\, h_{s,\vec{p},mM}(x) h^{\dagger}_{s,\vec{p},mM'}(x') \\ \left[\left(\nabla^{2}{}^{M}_{N} + m_A^2 \delta^M_N\right) - \nabla_N \nabla^M + \xi^{-1}\nabla^M \nabla_N\right]^2 \langle T A(x) A(x')\rangle. \end{aligned} \tag{E.4}$$

Multiplying by the appropriate state normalization $N = \sqrt{2p_{m0}m}$ allows us to write a variant of Eq. (E.4) that respects AdS isometries,

$$\begin{aligned} \langle s,\vec{p},m|s,\vec{p},m\rangle =& (i)^2 \int dx\, dx'\, \sqrt{g(x)g(x')}\, 2p_{m0} m h_{s,\vec{p},mM}(x) h^{\dagger}_{s,\vec{p},mM'}(x') \\ & \left[\left(\nabla^{2}{}^{M}_{N} + m_A^2 \delta^M_N\right) - \nabla_N \nabla^M + \xi^{-1}\nabla^M \nabla_N\right]^2 \langle T A(x) A(x')\rangle. \end{aligned} \tag{E.5}$$

This is the LSZ reduction formula that relates correlation functions of creation and annihilation operators to correlation functions of fields. The differential operator acting on the correlation function in the integrand of Eq. (E.5) generates contact terms that correspond to disconnected diagrams according to the Schwinger-Dyson equations.

The above steps can be repeated for any number of a's to arrive at, for instance, Eq. (3.44).

Appendix F

Review of Conformal Blocks and Partial Waves

As we have emphasized, there are two ways of looking at the (global) conformal blocks. One is purely algebraic — we simply apply the OPE algebra twice and use the orthogonality of the two point function. The other approach is to insert the identity operator and then organize the contribution to the four point function in terms of the representations under the conformal group, namely the spins and scaling dimensions of the exchanged operators. In both cases, the end result is the same — we can break up the structure of the correlation function into a sum over all conformal families of the theory. The dynamics are fully encoded in the coefficients of these operators while the blocks themselves only depend on the conformally invariant cross-ratios u and v. In other words, for a primary operator appearing in our theory, the conformal blocks tell us how much that primary and its descendants contribute to the four-point function.

Consider the four-point function of scalar fields $\phi_i \equiv \phi(x_i)$ with scaling dimensions Δ_i. We can decompose it into an overall conformally invariant structure multiplied by some generic function of the conformal cross ratios $G(u,v)$ as follows:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3 - \Delta_4}{2}} \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} G(u,v), \tag{F.1}$$

with $u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$. We can then express $G(u,v)$ as an expansion of functions (the conformal blocks) but noting that applying the OPE on the left hand side twice gives us

$$\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \langle \sum_{\tau,\ell} \lambda^{12}_{\tau,\ell} C(x_1 - x_2; \partial_2) \mathcal{O}_2 \sum_{\tau',\ell'} \lambda^{34}_{\tau',\ell'} C(x_3 - x_4; \partial_4) \mathcal{O}_4 \rangle \\
&= \sum_{\tau,\ell} \lambda^{12}_{\tau,\ell} \lambda^{34}_{\tau,\ell} C(x_1 - x_2; \partial_2) C(x_3 - x_4; \partial_4) \langle \mathcal{O}_2 \mathcal{O}_4 \rangle \\
&\equiv \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3 - \Delta_4}{2}} \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \sum_{\tau,\ell} \lambda^{12}_{\tau,\ell} \lambda^{34}_{\tau,\ell} G_{\tau,\ell}(u,v),
\end{aligned} \tag{F.2}$$

where we have used the fact that the two point function demands $\delta_{\tau,\tau'}\delta_{\ell,\ell'}$ due to orthogonality. The conformal blocks are denoted by $G_{\tau,\ell}(u,v)$ and the partial waves are defined as the blocks times additional coordinate-dependent prefactors:

$$W_{\Delta,\ell} \equiv \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3 - \Delta_4}{2}} \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} G_{\tau,\ell}(u,v). \tag{F.3}$$

Note then that one can obtain the function $G(u,v)$ from the conformal blocks via

$$G(u,v) = \sum_{\tau,\ell} \lambda^{12}_{\tau,\ell} \lambda^{34}_{\tau,\ell} G_{\tau,\ell}(u,v), \tag{F.4}$$

for the case of scalar fields. We sometimes abbreviate $G_{\tau,\ell}$ as $G_{\mathcal{O}}(u,v)$ and the sum therefore runs over $\mathcal{O}$.

Appendix G

Details and Definitions

G.1 Partial Wave Definitions

To account for possible dimension shifts, we define the scalar partial waves as a scalar part (χ) times the scalar block:

$$W_{\mathcal{O}}^{ij} \equiv \chi^{ij} G_{\mathcal{O}}^{ij}, \tag{G.1}$$

with

$$\chi^{ij} \equiv \frac{1}{P_{12}^{\frac{1}{2}(\Delta_1+i+\Delta_2+j)} P_{34}^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left(\frac{P_{24}}{P_{14}}\right)^{\frac{1}{2}(\Delta_1+i-\Delta_2-j)} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{1}{2}\Delta_{34}}, \tag{G.2}$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$. To obtain the partial wave in physical space, one may use the relation $P_{ij} = x_{ij}^2$.

G.2 Scalar Functions

The scalar functions result from writing derivative operators acting on z and $\bar{z}$ instead of P_1 and P_2. The point is that although u and v are the "physical" variables of the scalar blocks, their exact forms are hypergeometric functions in z and $\bar{z}$, which carry the implicit dependence on u and v. We can write

$$\frac{\partial G_{\mathcal{O}}}{\partial P_i^A} = \frac{\partial G}{\partial z}\left(\frac{\partial v}{\partial P_i^A}\frac{\partial z}{\partial v} + \frac{\partial u}{\partial P_i^A}\frac{\partial z}{\partial u}\right) + \frac{\partial G}{\partial \bar{z}}\left(\frac{\partial v}{\partial P_i^A}\frac{\partial \bar{z}}{\partial v} + \frac{\partial u}{\partial P_i^A}\frac{\partial \bar{z}}{\partial u}\right), \tag{G.3}$$

where the partial derivatives acting on $\bar{z}$ are somewhat complicated since the condition that $u = z\bar{z}$ and $v = (1-z)(1-\bar{z})$ lets us solve explicitly for z and $\bar{z}$ in terms of u and v (there are two solutions):

$$\left\{z \to \frac{1}{2}\left(-\sqrt{(u-v+1)^2-4u}+u-v+1\right),\right.$$
$$\left.\bar{z} \to \frac{1}{2}\left(\sqrt{(u-v+1)^2-4u}+u-v+1\right)\right\}, \tag{G.4}$$
$$\left\{z \to \frac{1}{2}\left(\sqrt{(u-v+1)^2-4u}+u-v+1\right),\right.$$
$$\left.\bar{z} \to \frac{1}{2}\left(-\sqrt{(u-v+1)^2-4u}+u-v+1\right)\right\}. \tag{G.5}$$

If we interpret z and $\bar{z}$ as coordinates then in the limit that $u, v \ll 1$, we find that in the first solution $z \to 0$ and $\bar{z} \to 1$ while for the second solution, $z \to 1$ and $\bar{z} \to 0$. Typically, in a four point function we can always do a rescaling of the external coordinates so we have $\langle\phi(0)\phi(z)\phi(1)\phi(\infty)\rangle$ which then implies $u \sim x_{12}^2 \sim z\bar{z}$. This means that which solution we pick is not of great importance since the correlation function only depends on the absolute distance between operators. Picking the second solution, we get

$$\mu(u,v) \equiv \frac{\partial z}{\partial u} = \frac{1}{2}\left(\frac{u-v-1}{\sqrt{(u-v+1)^2-4u}}+1\right), \tag{G.6}$$

$$\bar{\mu}(u,v) \equiv \frac{\partial \bar{z}}{\partial u} = \frac{1}{2}\left(\frac{-u+v+1}{\sqrt{(u-v+1)^2-4u}}+1\right), \tag{G.7}$$

$$\lambda(u,v) \equiv \frac{\partial z}{\partial v} = \frac{1}{2}\left(\frac{-u+v-1}{\sqrt{(u-v+1)^2-4u}}-1\right), \tag{G.8}$$

$$\bar{\lambda}(u,v) \equiv \frac{\partial \bar{z}}{\partial v} = \frac{1}{2}\left(\frac{u-v+1}{\sqrt{(u-v+1)^2-4u}}-1\right). \tag{G.9}$$

Appendix H

Applying Conservation

Recall that in position space, correlation functions involving currents are encoded into the z's (not to be confused with z and $\bar{z}$ that appear in the scalar blocks). Namely, for a correlator $f^{\mu_1\mu_2\cdots\mu_n}$, we have

$$\tilde{f}(x;z) \equiv f^{\mu_1\mu_2\cdots\mu_n}(x) z_{1,\mu_1} z_{2,\mu_2} \ldots z_{n,\mu_n}. \tag{H.1}$$

By lifting this to embedding space, we can recover the correlator in terms of the Z's and P's we have been using this whole time. However, it's easy to project back onto position space via the relations

$$Z_1 \cdot Z_2 \to z_1 \cdot z_2, \qquad P_1 \cdot P_2 \to -\frac{1}{2} x_{12}^2 \tag{H.2}$$

$$P_1 \cdot Z_2 \to z_2 \cdot x_{12}, \qquad P_2 \cdot Z_1 \to -z_1 \cdot x_{12}. \tag{H.3}$$

With this in mind, consider a two-point correlation function given by

$$\tilde{f}(x;z) = f^{\mu\nu}(x) z_{1,\mu} z_{2,\nu}. \tag{H.4}$$

So we see that the ∂_μ operator is implemented easily through the

$$\mathcal{D}_c \equiv \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial z} \tag{H.5}$$

operator. One question might be why can't we do it in embedding space? Well, the answer is in principle one could do that noting that we can transform the above operator into partial derivatives acting on P's and Z's. The unfortunate price to pay would be to keep track of tensors like $\frac{\partial Z^A}{\partial z}$. It is thus simpler to project onto physical space via the relations Eqs. (H.2) - (H.3) and then implement $\partial_x \cdot \partial_z$. It is easy to see that this argument generalizes quite readily to any n-point function as well, since evaluating the divergence at x_i, z_i will always amount to computing $\partial_\mu f^{\mu \cdots}$, modulo pre-factors of z that clearly cannot influence the conservation condition.

Appendix I

Full Result of Spin-1 Divergence

Let f_1 be coefficient function of $\partial^2 G$, f_2 for $\partial\bar{\partial}G$, and f_3 for $\bar{\partial}^2 G$. And for single derivatives, g_1 for ∂G and g_2 for $\bar{\partial}G$. Lastly, we denote h for the "finite" term. The divergence of the four point function is then written as

$$\begin{aligned}\partial\cdot G_1 &= f_1\partial^2 G^{10} + f_1'\partial^2 G^{01} + f_2\partial\bar{\partial}G^{10} + f_2'\partial\bar{\partial}G^{01} \\ &\quad + f_3\bar{\partial}^2 G^{10} + f_3'\bar{\partial}^2 G^{01} + g_1\partial G^{10} + g_1'\partial G^{01} + g_2\bar{\partial}G^{10} + g_2'\bar{\partial}G^{01} + hG^{10} + h'G^{01},\end{aligned} \tag{I.1}$$

where G_1 is the single current four point function and the G^{ij}'s are scalar blocks with dimension shifts corresponding to $\Delta_1 \to \Delta_1 + i$ and $\Delta_2 \to \Delta_2 + j$. The crux of our analysis is that at large Δ, k derivatives acting on the scalar blocks become $\Delta^k G$ but this involves new scalar functions. At large Δ, we found that

$$
\begin{aligned}
\partial \cdot G_1 \approx & (f_1 + f_1') \sum_{\Delta,l} \lambda^{12}_{\Delta,l} \lambda^{34}_{\Delta,l} F_{1,\Delta} G_{\Delta,l} \\
& + (f_2 + f_2') \sum_{\Delta,l} \lambda^{12}_{\Delta,l} \lambda^{34}_{\Delta,l} F_{2,\Delta} G_{\Delta,l} \\
& + (f_3 + f_3') \sum_{\Delta,l} \lambda^{12}_{\Delta,l} \lambda^{34}_{\Delta,l} F_{3,\Delta} G_{\Delta,l} \\
& + \text{ terms of order } \Delta \text{ and below,}
\end{aligned} \tag{I.2}
$$

where the functions $F_{i,\Delta}(z, \bar{z})$ came about from computing the double derivatives and taking the large Δ limit (for example, $F_{1,\Delta}$ would be the all the factors of z and $\bar{z}$ in front of the scalar blocks in Eq. (3.152) in the large Δ limit). To leading order in Δ, the functions $F_{i,\Delta}(z, \bar{z})$ are all proportional to Δ^2. If one uniformly rescales the partial wave coefficients such that $\lambda^{ij}_{\Delta,l} \to \Delta^{-1} \lambda^{ij}_{\Delta,l}$, then it is evident that one obtains scalar correlation functions to leading order. Here, we explicitly write down the f_i,

g_i, and h functions and F_1, F_2, F_3.

$$
\begin{aligned}
f_1 = c_1\chi^{10}\bigg\{ & -\frac{1}{2}x_{12}^2\bigg[2v\left(\frac{x_{13,\mu}}{x_{23}^2}-\frac{x_{14,\mu}}{x_{24}^2}\right)\left(\lambda^2\partial^\mu v+\lambda\mu\partial^\mu u\right) \\
& -\frac{2ux_{14,\mu}}{x_{24}^2}\left(\mu\lambda\partial^\mu v+\mu^2\partial^\mu u\right)\bigg] \\
& -\frac{2vx_{14,\mu}}{x_{24}^2}\left(\mu^2\partial^\mu u+\mu\lambda\partial^\mu v\right)\bigg\}, \qquad \text{(I.3)}\\
f_1' = c_2\chi^{01}\bigg\{ & -\frac{1}{2}x_{12}^2\bigg[2v\left(\frac{x_{14,\mu}}{x_{14}^2}-\frac{x_{13,\mu}}{x_{13}^2}\right)\left(\lambda^2\partial^\mu v+\mu\lambda\partial^\mu u\right) \\
& +2u\left(\frac{x_{12,\mu}}{x_{12}^2}-\frac{x_{13,\mu}}{x_{13}^2}\right)\left(\mu\lambda\partial^\mu v+\mu^2\partial^\mu u\right)\bigg]\bigg\}, \qquad \text{(I.4)}\\
f_2 = c_1\chi^{10}\bigg\{ & -\frac{1}{2}x_{12}^2\bigg[2v\left(\frac{x_{13,\mu}}{x_{23}^2}-\frac{x_{14,\mu}}{x_{24}^2}\right)\left(2\lambda\bar\lambda\partial^\mu v+\lambda\bar\mu\partial^\mu u+\bar\lambda\mu\partial^\mu u\right) \\
& -\frac{2ux_{14,\mu}}{x_{24}^2}\left(\mu\bar\lambda\partial^\mu v+\bar\mu\lambda\partial^\mu v+2\bar\mu\mu\partial^\mu u\right)\bigg] \\
& -\frac{2vx_{14,\mu}}{x_{24}^2}\left(2\mu\bar\mu\partial^\mu u+\mu\bar\lambda\partial^\mu v+\bar\mu\lambda\partial^\mu v\right)\bigg\}, \qquad \text{(I.5)}\\
f_2' = c_2\chi^{01}\bigg\{ & -\frac{1}{2}x_{12}^2\bigg[2v\left(\frac{x_{14,\mu}}{x_{14}^2}-\frac{x_{13,\mu}}{x_{13}^2}\right)\left(2\lambda\bar\lambda\partial^\mu v+\bar\mu\lambda\partial^\mu u+\mu\bar\lambda\partial^\mu u\right) \\
& +2u\left(\frac{x_{12,\mu}}{x_{12}^2}-\frac{x_{13,\mu}}{x_{13}^2}\right)\left(\mu\bar\lambda\partial^\mu v+\bar\mu\lambda\partial^\mu v+2\bar\mu\mu\partial^\mu u\right)\bigg]\bigg\}, \qquad \text{(I.6)}
\end{aligned}
$$

$$f_3 = f_1(\lambda\leftrightarrow\bar\lambda, \mu\leftrightarrow\bar\mu), \qquad \text{(I.7)}$$

$$f_3' = f_1'(\lambda\leftrightarrow\bar\lambda, \mu\leftrightarrow\bar\mu), \qquad \text{(I.8)}$$

$$
\begin{aligned}
g_1 =& c_1 \chi^{10} \Bigg\{ -\frac{1}{2} x_{12}^2 \Bigg[\frac{2\gamma}{x_{24}^2} x_{14,\mu} \left(\lambda \partial^\mu v + \mu \partial^\mu u \right) + 2v \partial_\mu \lambda \left(\frac{x_{13}^\mu}{x_{23}^2} - \frac{x_{14}^\mu}{x_{24}^2} \right) \\
& + 4 \left(\alpha \frac{x_{12}^\mu}{x_{12}^2} + (\gamma - k) \frac{x_{14}^\mu}{x_{14}^2} + k \frac{x_{13}^\mu}{x_{13}^2} \right) \left[\lambda v \left(\frac{x_{13,\mu}}{x_{23}^2} - \frac{x_{14,\mu}}{x_{24}^2} \right) - \mu u \frac{x_{14,\mu}}{x_{14}^2} \right] \\
& + 2\lambda \left[\partial^\mu v \left(\frac{x_{13,\mu}}{x_{23}^2} - \frac{x_{14,\mu}}{x_{24}^2} \right) + vd \left(\frac{1}{x_{23}^2} - \frac{1}{x_{24}^2} \right) \right] - 2u \frac{x_{14} \cdot \partial \mu}{x_{24}^2} - 2 \frac{\mu d v}{x_{24}^2} \Bigg] \\
& + 2v x_{12}^\mu \left[\left(\frac{x_{13,\mu}}{x_{23}^2} - \frac{x_{14,\mu}}{x_{24}^2} \right) \lambda - \frac{2x_{14,\mu}}{x_{24}^2} \mu \right] \\
& + 2 \left(\alpha + (\gamma - k) \frac{x_{12} \cdot x_{14}}{x_{14}^2} + k \frac{x_{12} \cdot x_{13}}{x_{13}^2} \right) \left[\lambda v \left(\frac{x_{13}^2}{x_{23}^2} - \frac{x_{14}^2}{x_{24}^2} \right) - \frac{x_{14}^2}{x_{24}^2} u \mu \right] \Bigg\},
\end{aligned} \tag{I.9}
$$

$$
\begin{aligned}
g_1' =& c_2 \chi^{01} \Bigg\{ -\frac{1}{2} x_{12}^2 \Bigg[-2 \left(\alpha \frac{x_{12,\mu}}{x_{12}^2} + \frac{x_{14,\mu}}{x_{14}^2} (\gamma' - k) + k \frac{x_{13,\mu}}{x_{13}^2} \right) \\
& \times \left(\lambda \partial^\mu v + \mu \partial^\mu u - 2u\mu \left(\frac{x_{12,\mu}}{x_{12}^2} - \frac{x_{13,\mu}}{x_{13}^2} \right) - 2v\lambda \left(\frac{x_{14,\mu}}{x_{14}^2} - \frac{x_{13,\mu}}{x_{13}^2} \right) \right) \Bigg] \\
& + 2\lambda \left[\partial^\mu v \left(\frac{x_{14,\mu}}{x_{14}^2} - \frac{x_{13,\mu}}{x_{13}^2} \right) + v(d-2) \left(\frac{1}{x_{14}^2} - \frac{1}{x_{13}^2} \right) \right] \\
& + 2\mu \left[\partial^\mu u \left(\frac{x_{12,\mu}}{x_{12}^2} - \frac{x_{13,\mu}}{x_{13}^2} \right) + u(d-2) \left(\frac{1}{x_{12}^2} - \frac{1}{x_{13}^2} \right) \right] \\
& - (\alpha + \gamma') \left(\lambda x_{12} \cdot \partial v + \mu x_{12} \cdot \partial u \right) \\
& + 2\lambda v \left(\frac{x_{12} \cdot x_{13}}{x_{13}^2} + \frac{x_{12} \cdot x_{14}}{x_{14}^2} \right) + 2\mu u \left(\frac{x_{12} \cdot x_{13}}{x_{13}^2} - 1 \right) \Bigg\},
\end{aligned} \tag{I.10}
$$

$$ \tag{I.11} $$

$$g_2 = g_1(\lambda \leftrightarrow \bar{\lambda}, \mu \leftrightarrow \bar{\mu}), \tag{I.12}$$

$$g_2' = g_1'(\lambda \leftrightarrow \bar{\lambda}, \mu \leftrightarrow \bar{\mu}), \tag{I.13}$$

$$\begin{aligned} h =& c_1 \chi^{10} \left\{ \frac{2\gamma}{x_{24}^2} \left[d - x_{12} \cdot x_{14} + 2 \left(\alpha \frac{x_{12} \cdot x_{14}}{x_{12}^2} + \gamma - k + k \frac{x_{14} \cdot x_{13}}{x_{13}^2} \right) \right.\right. \\ &\left.\left. + x_{14}^2 \left(\alpha + (\gamma - k) \frac{x_{12} \cdot x_{14}}{x_{14}^2} + k \frac{x_{12} \cdot x_{13}}{x_{13}^2} \right) \right] \right\}, \end{aligned} \tag{I.14}$$

$$\begin{aligned} h' =& c_2 \chi^{01} \left\{ 2(1 - \alpha - \gamma') \left(\alpha + \frac{x_{12} \cdot x_{14}}{x_{14}^2} (\gamma' + k) + k \frac{x_{12} \cdot x_{13}}{x_{13}^2} \right) \right. \\ & + x_{12}^2 \left[2 \left(\alpha \frac{x_{12,\mu}}{x_{12}^2} + \frac{x_{14,\mu}}{x_{14}^2} (\gamma' - k) + k \frac{x_{13,\mu}}{x_{13}^2} \right)^2 + (d-2) \left(\frac{\alpha}{x_{12}^2} + \frac{\gamma' - k}{x_{14}^2} + \frac{k}{x_{13}^2} \right) \right] \\ & \left. - d(\alpha + \gamma') \right\}, \end{aligned} \tag{I.15}$$

where c_1 and c_2 are arbitrary, independent coefficients, d refers to CFT$_d$, u and v are the conformally invariant cross ratios, $\partial \equiv \partial_{x_1,\mu}$, and χ is the partial wave pre-factor. If the current is conserved, one may relate c_1 and c_2 (see methods for conserved tensors in [2]). The quantities α, γ, γ', k, are

$$\alpha \equiv \frac{\Delta_1 + \Delta_2 + 1}{2}, \tag{I.16}$$

$$\gamma \equiv \frac{\Delta_1 - \Delta_2 + 1}{2}, \tag{I.17}$$

$$\gamma' \equiv \frac{\Delta_1 - \Delta_2 - 1}{2}, \tag{I.18}$$

$$k \equiv \frac{\Delta_3 - \Delta_4}{2}. \tag{I.19}$$

Finally, the functions μ, $\bar{\mu}$, λ, and $\bar{\lambda}$ are defined in Eqs. (G.6) - (G.9).

Large Δ Limit:

$$F_1 = \Delta^2 \left\{ \frac{1}{16z} - \frac{1}{8z\sqrt{z}} - \frac{1}{4z^2(z-\bar{z})} \left[\frac{1}{4\sqrt{z}} (z+\bar{z}) + (\bar{z}-z) \right] \right\} + \ldots , \tag{I.20}$$

$$F_2 = \Delta^2 \left(\frac{1}{4z\bar{z}} + \frac{1}{8z\sqrt{\bar{z}}} + \frac{1}{16z\bar{z}\sqrt{z}} + \frac{1}{32z\sqrt{z\bar{z}}} \right) + \ldots \tag{I.21}$$

$$F_3 = \Delta^2 \left(\frac{5}{16z\bar{z}} + \frac{1}{4z\sqrt{\bar{z}}} \right) + \ldots , \tag{I.22}$$

where the $\ldots$ indicate terms that are order Δ and below.

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Vita

Sean Cantrell was born to John H. Cantrell and Davie Sue Cantrell in Norfolk, Virginia, on April 7, 1987. He first began conducting research using the atomic force microscope (AFM) at NASA Langley Research Center (LaRC) in high school through the New Horizons Governor's School Mentorship program in Hampton, Virginia. He graduated with highest honors with a B.S. in Physics and B.A. in Mathematics at the University of Virginia (UVa) in 2009. During his tenure at UVa, he began his studies in particle theory under Professor Peter Arnold's mentorship while additionally continuing to pursue research at LaRC using the AFM. His efforts earned him a Goldwater Scholarship and the titles Science Scholar and Echols Scholar within the UVa community. While at Johns Hopkins, he worked on the side as a consultant for Chicago-based startup Nanosonix.

In 2013, he cofounded a company that deals broadly with nonlinear metrology with particular applications to atomic force microscopy and micro electro-mechanical devices. Sean will continue working with his company after graduation.